Quantum charged fields in (1+1) Rindler space.

Cl. Gabriel[*][†]and Ph. Spindel[‡]

*Mécanique et Gravitation, Université de Mons-Hainaut,
6, avenue du Champ de Mars, B-7000 Mons, Belgium*

Abstract

We study, using Rindler coordinates, the quantization of a charged scalar field interacting with a constant (Poincaré invariant), external, electric field in (1+1) dimensionnal flatspace: our main motivation is pedagogy. We illustrate in this framework the equivalence between various approaches to field quantization commonly used in the framework of curved backgrounds. First we establish the expression of the Schwinger vacuum decay rate, using the operator formalism. Then we rederive it in the framework of the Feynman path integral method. Our analysis reinforces the conjecture which identifies the zero winding sector of the Minkowski propagator with the Rindler propagator. Moreover we compute the expression of the Unruh's modes that allow to make connection between Minkowskian and Rindlerian quantization scheme by purely algebraic relations. We use these modes to study the physics of a charged two level detector moving in an electric field whose transitions are due to the exchange of charged quanta. In the limit where the Schwinger pair production mechanism of the exchanged quanta becomes negligible we recover the Boltzman equilibrium ratio for the population of the levels of the detector. Finally we explicitly show how the detector can be taken as the large mass and charge limit of an interacting fields system.

---

[*]Aspirant du F.N.R.S.
[†]e-mail : gabriel@sun1.umh.ac.be
[‡]e-mail : spindel@umh.ac.be

1   Introduction

Shortly after Hawking's discovery of black hole evaporation [1], Unruh [2] showed that a uniformly accelerated detector moving in flat space perceives the Minkowski vacuum to be thermally populated at temperature $T_U = a/2\pi$. On the other hand, Heisenberg and Euler [3] showed in 1936 that the vacuum state of a charged quantum field interacting with a static electric field is unstable and decays into pairs. In 1951 Schwinger [4]gave the expression of this decay rate, using the technique known today as the Schwinger proper time representation of the functional integral. One can use the same formalism to describe these two phenomena, wherein the second occurs in the secure framework of usual quantum field in flat space. In this way we have a physically relevant model whose interpretation is unambiguous and which allows to exemplify several formal developments and check their validity. In this paper we mainly study, using Rindler coordinates on the (1+1) dimensional Minkowski space, the quantization of a charged field interacting with a background, constant, electric field E.The main motivation of this work is pedagogical: we hope to illustrate (and check) in the framework of an exactly (but non trivial) solvable model several conjectures that often are taken for granted in quantum field theory on curved spaces. In the framework of a charged quantum field interacting with an external constant electric field, we explicitly show that, the formal evaluation of vacuum decay rate based on an expression of $\langle 0, out|0, in \rangle$ as a functional integral on quantum fields leads to results in accord with standard calculations,even in Rindler coordinates, where the integrals are no more gaussian. This supports the use of a similar approach for more complicated problems, for which no exact solution is known. We also illustrate several aspects of physics that we believe to be important; for instance: 1) the occurrence of a boundary effect, whose analog in the framework of the physics of the Boulware vacuum in Schwarzschild geometry erases the Hawking radiation; 2) the validity of the thermodynamical equilibrium relation in (and only in) the limit of heavy systems; 3) the significance of the "rindlerian energy" balance 4) the contribution of the Schwinger-like and Unruh-like vacuum fluctuations on the transitions of the detector, similar to the Schwinger and Hawking mechanisms of charged black-holes evaporation. Our paper is organized as follows. Section **1** is just a summary of the main body of the text. In section **2** we recall the definition of Rindler coordinates and classify the classical trajectories of charged particles in a constant electric field. The results of this analysis are intensively exploited when we interpret the subsequent results. In section **3** we recall the derivation of the Schwinger effect in the framework of the usual quantization (using the creation and annihilation operators formalism in Lorentzian coordinates) of a charged scalar field interacting with an external constant electric field. Then, in section **4**, we perform the same work in Rindler coordinates. The results differ from those obtained using global coordinates by boundary terms. We interpret this difference as the manifestation of the same mechanism leading to the difference between the Boulware's and Unruh's vacua in the framework of black hole physics. Then, in section **5**, we reconsider the problem in the light of the functional path integral method, and illustrate the validity of the standard approximation scheme (saddle point – W.K.B. approximation) used to evaluate functional integrals, by comparing their predictions to the results obtained previously; we show that both methods (fortunately) agree up to boundary terms. In the same way, we also reinforce the interpretation of the usual minkowskian propagator as a sum over winding Rindlerian propagators [5]. Indeed using an expression for what we believed to be the exact Rindler propagator to compute the rate of particle creation from the vacuum, we shall recover at one

and the same time its leading term and the boundary correction terms. We also compute, in section **6**, the Rindlerian particle content of the Minkowskian vacua. In section **7**, we study the behavior of accelerated charged detectors. We show that the detector transition (in the limit where the charge of the exchanged quanta vanishes) are dominated by the induced transition of the detector with preexisting vacuum fluctuations that maximally overlap the detector world line. We also establish precisely how to interpret a two level charged detector as a limiting case of large mass M and charge Q, but of finite acceleration $Q\,E/M$, of an interacting field [6, 7] system. Finally, some mathematical appendices provide the technicalities underlying the results of the main text.

2   Classical trajectories in Rindler coordinates

We recall that Rindler coordinates divide Minkowski space into four patches, denoted hereafter by the labels $\underline{\mathbf{R}}$(ight), $\underline{\mathbf{L}}$(eft), $\underline{\mathbf{P}}$(ast) and $\underline{\mathbf{F}}$(uture). On each of these patches the coordinate transformations between Minkowski (t, z) and Rindler (τ, ξ) coordinates are given by

$$\mathbf{R}\begin{cases} t &=& a^{-1}e^{a\,\xi_R}\sinh a\tau_R \\ z &=& a^{-1}e^{a\,\xi_R}\cosh a\tau_R \end{cases} (z>0, |t|<z) \qquad \mathbf{L}\begin{cases} t &=& -a^{-1}e^{a\,\xi_L}\sinh a\tau_L \\ z &=& -a^{-1}e^{a\,\xi_L}\cosh a\tau_L \end{cases} (z<0, |t|<z)$$

$$\mathbf{P}\begin{cases} t &=& -a^{-1}e^{a\,\xi_P}\cosh a\tau_P \\ z &=& -a^{-1}e^{a\,\xi_P}\sinh a\tau_P \end{cases} (t<0, |z|<|t|) \qquad \mathbf{F}\begin{cases} t &=& a^{-1}e^{a\,\xi_F}\cosh a\tau_F \\ z &=& a^{-1}e^{a\,\xi_F}\sinh a\tau_F \end{cases} (t>0, |z|<t)$$

$$(2.1)$$

On each of these quadrants, the respective Rindler coordinates run from $-\infty$ to ∞. In the following, from time to time, we shall also make use of another standard Rindler coordinate :

$$\rho = a^{-1}e^{[2\,a\,\xi]} \qquad ,$$

which is obviously always positive.

Let us emphasize that on $\mathbf{R}$ and $\mathbf{L}$ the vector field ∂_τ is timelike, pointing respectively to the future and the past while on $\mathbf{F}$ and $\mathbf{P}$ it is ∂_ξ that provides us timelike directions, pointing toward the future on $\mathbf{F}$ and the past on $\mathbf{P}$. In the following, we will be led to consider the various components of the boundaries of these patches. In two dimensions, the boundary of the full Minkowski space consists into four points: the future and past timelike infinities, denoted by $i^+ and\ i^-$, the spacelike left and right infinities, i_L^0 and i_R^0, and the four components of the null infinity: $\mathcal{I}_R^+$, $\mathcal{I}_L^+$, $\mathcal{I}_R^-$ and $\mathcal{I}_L^-$. Moreover, the light cone issuing from the origin O defines the acceleration horizons of stationary Rindler's observers, whose trajectory equations are $\xi = C^{te}$ on $\mathbf{R}$, $\mathbf{L}$ and $\tau = C^{te}$ on $\mathbf{P}$ and $\mathbf{F}$. It splits into four branches (constituted by the null rays emerging from its vertex O or ending on it): $\mathcal{H}_R^+$, $\mathcal{H}_R^-$, $\mathcal{H}_L^+$ and $\mathcal{H}_L^-$. They constitute parts of the boundary of the different Rindler's patches. The other components of their "boundaries" [1] are the points i^+, i^-, i_L^0, i_R^0 and the eight pieces of the null infinity components denoted

---

[1]Between quotation marks because they are not really boundaries of regions in Minlowski space but boundaries of the conformal Carter-Penrose diagram. In the rest of the paper, we shall not make explicit this distinction.

by $\mathcal{I}^+_{R,F}$ (the future null infinity boundary of the **R** quadrant), $\mathcal{I}^+_{F,R}$, $\mathcal{I}^+_{L,F}$, ..., $\mathcal{I}^-_{P,R}$. All these geometrical considerations are summarized on fig. (**1**) which represents the well-known conformal Carter-Penrose diagram of (1+1) dimensional Minkowski space-time.

A constant electric field, which in flat coordinates is given by $\mathcal{F} = E dz \wedge dt$ with $E = Cst$, reads as [2] $\mathcal{F} = \epsilon E e^{2a\,\xi} d\xi \wedge d\tau$ with $\epsilon = +1$ on **R**, **L** and $\epsilon = -1$ on **P** and **F**. This field is invariant with respect to the Poincaré group, acting on the (1+1) Minkowski space. It can be derived from the potential:

$$A = \frac{E}{2}(z dt - t dz) = \epsilon \frac{E}{2} e^{2a\,\xi} a^{-1} d\tau \qquad . \tag{2.2}$$

Classically, it accelerates positively charged particles towards the right ($z > 0$), negatively charged particles towards the left ($z < 0$). The Hamiltonian describing these classical motions is

$$H_{cl} = -\frac{\epsilon e^{-2a\,\xi}}{2m}\left[\left(p_\tau - \frac{\epsilon q E e^{2a\,\xi}}{2a}\right)^2 - p_\xi^2\right] \qquad , \tag{2.3}$$

the momentum p_τ is a constant of motion:

$$p_\tau = \epsilon m e^{2a\,\xi}\left(\frac{qE}{2ma} - \dot{\tau}\right) = -\omega \tag{2.4}$$

and, once we impose the evolution parameter to be the proper time[3] along the trajectory, the mass shell value of the Hamiltonian is fixed : $H_{cl} = -m/2$, which determines p_ξ:

$$p_\xi^2 = -\epsilon m^2 e^{2a\,\xi} + \left(\omega + \epsilon\frac{qE}{2a}e^{2a\,\xi}\right)^2 \qquad . \tag{2.5}$$

On **R** and **L**, where $\epsilon = 1$, $\dot{\tau}$ is of constant sign on timelike trajectories ($\dot{\tau}^2 = \dot{\xi}^2 + e^{-2a\xi}$), while $p_\xi(= +m\dot{\xi}e^{2a\,\xi})$ vanishes at the turning points ξ_+ and ξ_-:

$$\xi_\pm = \frac{1}{a}\log\left\{\left(\frac{ma}{qE}\right)\left[1 \pm (1 - 2\Omega)^{1/2}\right]\right\} \tag{2.6}$$

when $\Omega \equiv \omega qE/m^2 a$ is less than $\frac{1}{2}$. At these turning points:

$$\dot{\tau}_\pm = \frac{qE}{2am\Omega}\left[1 \mp (1 - 2\Omega)^{1/2}\right] \qquad ; \tag{2.7}$$

showing that, for $\Omega < 0$, $\dot{\tau}_-$ is negative on the trajectories for which $\xi < \xi_-$. The interpretation of these latter is obtained by noticing that particles moving forward in time (with respect to the global time orientation of Minkowski space), move with $\dot{\tau} > 0$ on **R** and $\dot{\tau} < 0$ on **L**, while anti-particles go in the opposite way.

Finally, the acceleration at the turning points is:

$$\ddot{\xi} = \pm a e^{-2a\,\xi_\pm}(1 - 2\Omega)^{1/2}. \tag{2.8}$$

---

[2] When no confusion arises we do not make explicit on which patch the Rindler's coordinates are considered.
[3] A dot indicating a derivative with respect to it.

Classical trajectories are branches of hyperbola (see fig. (**2**)). The geometrical significance of the constant Ω is obtained by noticing that :

$$\Delta^2 = \left(\frac{m}{qE}\right)^2 [1 - 2\,\Omega] \tag{2.9}$$

is the squared invariant distance from the origin O to the center of the hyperbolic trajectory (see fig. (**2**)).

On **R** and **L**, according the values of Ω, we distinguish three classes of trajectories.

- Those, on which $\Omega > \frac{1}{2}$. They have no turning points and correspond to motion with ξ varying between from $-\infty$ to $+\infty$, with $\dot{\xi} > 0$. The centers of these hyperbolas lie in the sector **P** or **F**. Moreover, $\dot{\tau} > 0$, so these trajectories describe particles in the **R** sector and anti-particles in **L**; more precisely, they describe:

  - On **R**, particles [1] (the numbers between brackets refer to the trajectories of fig. (**1**)) entering from **P** (with $\dot{\xi} > 0$), across the past horizon $\mathcal{H}_R^-$ with $\dot{\xi} > 0$ and running towards the null infinity component $\mathcal{I}_{R,F}^+$ or [2] coming from $\mathcal{I}_{R,P}^-$ (with $\dot{\xi} < 0$) and leaving **R** via $\mathcal{H}_R^+$,

  - On **L**, anti-particles (particles going backward in time) moving between the infinity and horizon components $\mathcal{I}_{L,F}^+$ and $\mathcal{H}_L^-$ with $\dot{\xi} < 0$, or between $\mathcal{I}_{L,P}^-$ and $\mathcal{H}_L^+$ with $\dot{\xi} < 0$.

- When $\frac{1}{2} > \Omega > 0$ the trajectories present a turning point but $\dot{\tau}$ is still positive, thus they describe motions of particles on **R** and anti-particles on **L** . The center of the hyperbolas are now located in the **R** or **L** sectors, but are "inside" the hyperbolic trajectory $\Delta^2 = \left(\frac{m}{qE}\right)^2$. In the **R** sector, the trajectories [3] with turning point ξ_- describe particles, which enter in **R** by crossing $\mathcal{H}_R^-$ with $\dot{\xi} > 0$, pass through maximum ξ at the turning point, and quit **R** by crossing $\mathcal{H}_R^+$. The trajectories [4] with turning point ξ_+ connect $\mathcal{I}_{R,P}^-$ and $\mathcal{I}_{R,F}^+$, and pass through a minimum ξ at the turning.

- When $\Omega < 0$, the center of the hyperbolas are located "outside" the limit hyperbola $\Delta^2 = \left(\frac{m}{qE}\right)^2$. Trajectories [4'] corresponding to the turning points ξ_+ describe also particles (because $\dot{\tau} > 0$) connecting $\mathcal{I}_{R,P}^-$ and $\mathcal{I}_{R,F}^+$. They differ from those with $\frac{1}{2} > \Omega > 0$ by the fact that the "partner" trajectories [5] (the other branches of the hyperbolas) intersect the **R** sector. Finally, trajectories associated to the turning points ξ_- describe anti-particles entering **R** by crossing $\mathcal{H}_R^-$ and leaving by $\mathcal{H}_R^+$.

On the quadrants **P** and **F**, ($\epsilon = -1$), the sign of the momentum $p_\xi = -m\dot{\xi}e^{2a\,\xi}$ remains constant along the trajectories and allows us to distinguish particle trajectories from anti-particle ones. Particles move with $\dot{\xi} > 0$ on **F** and $\dot{\xi} < 0$ on **P**. At null infinity $\dot{\tau}_\infty = \pm\dot{\xi}_\infty = qE/2ma > 0$ in accordance with the fact that particles [6] enter into **P** from the right past null infinity $\mathcal{I}_{R,P}^-$ (with $d\tau > 0$, $d\lambda > 0$ and $\dot{\xi}_\infty < 0$) while anti-particles [7] enter from the left past null infinity $\mathcal{I}_{L,P}^-$ (with $d\tau < 0$, $d\lambda < 0$) and $\dot{\xi}_\infty > 0$). Similarly, on **F**, particles [8] go asymptotically to the right component of the null infinity ($\mathcal{I}_{F,R}^+$) (with $d\tau$ and $d\lambda$ positive) while anti-particles [9]

go near $\mathcal{I}_{F,L}^+$ (with $d\tau$ and $d\lambda$ negative and $d\xi > 0$). On the horizons, $\dot{\tau}$ and Ω have opposite signs. So, the variable τ varies monotonically when $\Omega < 0$ but when $\Omega > 0$, $\dot{\tau}$ vanishes at $\xi = \frac{1}{2a}\log[\frac{2a\omega}{qE}]$. When $\Omega > 0$, on the **P** sector, the corresponding (particle) trajectories [6] connect $\mathcal{I}_{P,R}^-$ to $\mathcal{H}_R^-$ and the (anti-particle) trajectories $\mathcal{I}_{P,L}^-$ to $\mathcal{H}_L^-$. On the **F** sector, they describe anti-particles connecting $\mathcal{I}_{F,L}^+$ to $\mathcal{H}_L^+$ or particles between $\mathcal{H}_R^+$ and $\mathcal{I}_{F,R}^+$. The world lines of typical trajectories are depicted on fig. (**2**). Note that, due to the conformal character of this picture, the neighborhood of infinity is contracted into a finite region and the asymptotic tangency of trajectories with their asymptotic null rays is no more explicitly apparent, except in the fact that both reach the same points at infinity.

3   Field quantization and Schwinger effect

The quantization of a charged scalar field in an external background electric field is straight-forward. It consists into three main steps:

1) Introduce a covariant derivative $\mathcal{D}_\mu = \partial_\mu - iqA_\mu$ and determine a complete set of solutions of the wave equation

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi = m^2 \phi \qquad , \tag{3.1}$$

2) Separate the modes into two classes $\{\phi_A\}$ associated to particles and $\{\psi_{\tilde{A}}^*\}$ associated to anti-particle, normalize them using the scalar product built from the usual bilinear current

$$J_\mu(\phi_A, \phi_{A'}) = -i\phi_A^* \overset{\leftrightarrow}{\mathcal{D}}_\mu \phi_{A'} \tag{3.2}$$

integrated on appropriated Cauchy's surface Σ:

$$\langle \phi_A, \phi_{A'} \rangle = \int_\Sigma J_\mu(\phi_A, \phi_{A'}) dx^\mu \quad , \tag{3.3}$$

and define the quantum field

$$\hat{\Phi} = \sum_A a(A)\phi_A + \sum_{\tilde{A}} b^+(\tilde{A})\psi_{\tilde{A}}^* \qquad , \tag{3.4}$$

whose amplitudes satisfy usual commutation relations :

$$\begin{array}{ll} [a(A), a^+(A')] = \delta_{AA'} & [b(\tilde{A}), b^+(\tilde{A}')] = \delta_{\tilde{A}\tilde{A}'} \\ [a(A), b(\tilde{A})] = 0 & \text{etc ...} \end{array} \tag{3.5}$$

3) Build the Fock space from the vacuum state $|0\rangle$ defined by

$$a(A)|0\rangle = 0 = b(\tilde{A})|0\rangle \quad . \tag{3.6}$$

6

The interesting physics, the well known Schwinger phenomenon [4], results from the scattering of the waves on the external field. In the framework of second quantized field theory, a mode describing an incoming particle evolves into a mode associated to the superposition of outgoing particles and anti-particles [8]. Let us briefly recall how this mechanism works in Minkowskian coordinates. In the gauge (2.2), equation (3.1) reads:

$$\left[(\partial_z + \frac{iqEt}{2})^2 - (\partial_t - \frac{iqEz}{2})^2\right]\phi = m^2\phi \qquad , \qquad (3.7)$$

but as in the gauge $A = Ezdt$, the Dalembertian operator and ∂_t commute, the solutions of equation (3.7) can be expressed as superposition of the modes $e^{-i\sigma t}e^{i\frac{qE}{2}tz}\varphi_\sigma(z)$ where the functions $\varphi_\sigma(z)$ obey the equation:

$$\left[\partial_z^2 + (\sigma + qEz)^2\right]\varphi_\sigma(z) = m^2\varphi_\sigma(z). \qquad (3.8)$$

whose general solution can be expressed as a linear combination of parabolic cylinder functions $D_\nu[\zeta]$; (see for instance refs [5, 8] for a detailed discussion of the solutions of this equation and their physical interpretation). The modes describing incoming (anti-)particles are expressed in terms of parabolic cylinder functions (whose integral representation will be given in section **6**).

$$\phi_\sigma^{p\,in} = \frac{1}{M}D_{i\frac{m^2}{2qE}-\frac{1}{2}}[\zeta]\,e^{-i\sigma t}e^{i\frac{qE}{2}tz} \qquad , \qquad \phi_\sigma^{a\,in*} = \frac{e^{\frac{3i\pi}{4}}}{M}D_{-i\frac{m^2}{2qE}-\frac{1}{2}}[-\zeta^*]\,e^{-i\sigma t}e^{i\frac{qE}{2}tz} \qquad (3.9)$$

with $\zeta = e^{-\frac{3i\pi}{4}}\sqrt{2qE}(z+\frac{\sigma}{qE})$. Equivalently, modes corresponding to outgoing (anti)-particles are given by $\phi_\sigma^{p\,out}[t,z] = (\phi_\sigma^{p\,in}[-t,z])^*$ and $\phi_\sigma^{a\,out}[t,z] = (\phi_\sigma^{a\,in}[-t,z])^*$. The constant M has been fixed in order that these modes are normalized as follows:

$$\begin{aligned}
\langle\phi_\sigma^{p\,in},\phi_{\sigma'}^{p\,in}\rangle &= \langle\phi_\sigma^{p\,out},\phi_{\sigma'}^{p\,out}\rangle = \delta(\sigma-\sigma') \qquad , \\
\langle\phi_\sigma^{a\,in},\phi_{\sigma'}^{a\,in}\rangle &= \langle\phi_\sigma^{a\,out},\phi_{\sigma'}^{a\,out}\rangle = -\delta(\sigma-\sigma') \qquad , \\
\langle\phi_\sigma^{p\,in},\varphi_{\sigma'}^{a\,in*}\rangle &= \langle\phi_\sigma^{p\,out},\phi_{\sigma'}^{a\,out*}\rangle = 0 \qquad .
\end{aligned} \qquad (3.10)$$

These two basis are related by the Bogoljubov transformation:

$$\begin{aligned}
\phi_\sigma^{p\,out} &= \gamma\,\phi_\sigma^{p\,in} - \delta^*\phi_\sigma^{a\,in*} \qquad , \\
\phi_\sigma^{a\,out*} &= \gamma^*\phi_\sigma^{a\,in*} - \delta\,\phi_\sigma^{p\,in}
\end{aligned} \qquad (3.11)$$

with:

$$\gamma = \frac{\sqrt{2\pi}}{\Gamma[\frac{1}{2}+i\frac{m^2}{2qE}]}e^{-\frac{\pi m^2}{4qE}}e^{i\frac{\pi}{2}} \qquad , \qquad \delta = e^{-\frac{\pi m^2}{2qE}}e^{i\frac{\pi}{2}} \qquad ,$$

coefficients that verify the charge conservation relation $|\gamma|^2 - |\delta|^2 = 1$.
The *in* and *out* vacua are related by:

$$|0,Mink,in\rangle = N^{-1/2}e^{\frac{\delta^*}{\gamma}\sum_\sigma a_\sigma^{\dagger\,out}b_\sigma^{\dagger\,out}}|0,Mink,out\rangle \qquad (3.12)$$

7

where we have fixed an arbitrary phase by setting $N = (\prod_\sigma |\gamma|^2)$. Equation (3.12) show explicitly that the *in*-vacuum is filled with *out*-particles. This is the content of the Schwinger effect. Computing the probability of persistence of the vacuum, we obtain:

$$
\begin{aligned}
|<0, out|0, in>|^2 &= \exp\left[-\sum_\sigma \ln\left(1 + |\delta|^2\right)\right] \\
&= e^{-L\,T\,\Gamma}
\end{aligned}
\tag{3.13}
$$

i.e., in the limit of large space-time volume $L \times T$, the rate:

$$
\Gamma = \frac{qE}{2\,\pi} \ln\left(1 + e^{-\frac{\pi m^2}{qE}}\right)
\tag{3.14}
$$

of pair creation by unit of space-time volume. The link between the number of modes $\sum_\sigma$ and the factor L T $qE/2\,\pi$ is obtained as follows [9, 8]. Classically, the trajectories of charged particles in a constant electric field are hyperbolas. Each is characterized by the location of its center and its radius $= m/qE$. Quantum mechanically, it appears that it is in the neighborhood of the classical center, in a region of space-time extension of the order of $(m/qE) \times (m/qE)$ that a wave packet describing an incoming particle gives birth to an anti-particle. Accordingly, the number of modes relevant to the pair creation mechanism in a given space-time domain, of dimensions T×L, are those who centers belong to this domain. The density of modes of frequency σ in the time-t interval T is $(T/2\,\pi)\,d\sigma$. The centers of the relevant modes are located at the points $z = -\sigma/qE$ belonging to the space interval L=$z_1 - z_2$. Therefore the values of σ, which contribute to the sum, are those include in the interval $[-qE\,z_1,\ -qE\,z_2]$ so that in the limit of large space-time volume (in order that the boundary effects that are not taken into account in this counting of modes become negligible) we obtain $\sum_\sigma = (T/2\,\pi)\int_{qEz_1}^{qEz_2} d\sigma = T\,L\,(qE/2\,\pi)$.

4   Rindlerian field quantization

In Rindler coordinates, the wave equation (3.1) becomes:

$$
\left[-e^{-2a\xi}(\partial_\tau^2 - \partial_\xi^2) + i\frac{\epsilon qE}{a}\partial_\tau + \left(\frac{qE}{2}\right)^2 a^{-2}e^{2a\xi}\right]\phi = \epsilon m^2\phi
\tag{4.1}
$$

As ∂_τ commutes with the Dalembertian operator, the general solution of the wave equation can be expressed as a superposition of modes $\phi_\omega = (e^{-i\omega\tau}/\sqrt{2\pi})\mathcal{F}_\omega(\xi)$. The functions $\mathcal{F}_\omega(\xi)$ obey the second order differential equation:

$$
\left[\frac{d^2}{d\xi^2} + \left(\omega + \epsilon\frac{qE}{2a}e^{2a\xi}\right)^2 - \epsilon m^2 e^{2a\xi}\right]\mathcal{F}_\omega(\xi) = 0
\tag{4.2}
$$

The asymptotic behaviors of the solutions of this equation[4] are easily obtained by the standard W.K.B. technique. The modes ϕ_ω[5] can be expressed as:

$$\phi_\omega \approx \left\{ C_+^\omega \exp +i\left(\epsilon \frac{qE}{4a^2} e^{2a\xi} + (\omega - \frac{m^2 a}{qE})\xi \right) \right.$$
$$\left. + C_-^\omega \exp -i\left(\epsilon \frac{qE}{4a^2} e^{2a\xi} + (\omega - \frac{m^2 a}{qE})\xi \right) \right\} \left(\frac{2a}{qE} \right)^{1/2} e^{-a\xi} \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} \qquad (4.3)$$

when $\xi \approx +\infty$ i.e. near $\mathcal{I}^+$ and $\mathcal{I}^-$, and

$$\phi_\omega \approx \frac{1}{\sqrt{2|\omega|}} \left\{ D_+^\omega \exp i \left[\omega\,\xi \right] + D_-^\omega \exp -i \left[\omega\,\xi \right] \right\} \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} \qquad (4.4)$$

when $\xi \approx -\infty$ i.e. near the horizons. More precisely, near $\mathcal{I}_R^+$ and $\mathcal{I}_L^-$, whose points are labeled by $u = \tau - \xi$ coordinates, the relevant part of a mode is given by the weight of its $\exp(-i\omega u)$ component, i.e. its C_+^ω coefficient. Similarly, near $\mathcal{I}_R^-$ and $\mathcal{I}_L^+$, it is its $\exp(-i\omega v)$ part, i.e. its C_-^ω coefficient that governs it. Near the horizons, there are the components $D_-^\omega \exp(-i\omega v)$ and $D_+^\omega \exp(-i\omega u)$ that become relevant according to whether we are close to $\mathcal{H}_R^-$ or $\mathcal{H}_L^+$ on one hand or near $\mathcal{H}_R^-$ or $\mathcal{H}_L^+$ on the other. Of course all these different weights (C_+^ω, C_-^ω, D_-^ω, D_+^ω) are not independent. They are related by charge conservation. The scalar product (3.2, 3.3) is defined on the Rindler patches **R** and **L** by choosing as Cauchy's surfaces sections $\tau = C^{te}$. On such sections the scalar product of the modes we are considering reads as:

$$< \phi_\omega, \phi_{\omega'} >_{R,L} = \theta_{R,L} \int_{-\infty}^{+\infty} (\omega + \omega' + \frac{qE}{a} e^{2a\xi}) \mathcal{F}_{\omega'}^* \mathcal{F}_\omega \; d\xi$$
$$\equiv Q_\omega^{R,L} \delta(\omega - \omega') \qquad . \qquad (4.5)$$

The prefactor $\theta_{R,L}$ is given by the relative sign of the orientation of the vector field ∂_τ with respect to the global time orientation: $\theta_R = 1, \theta_L = -1$. Using the asymptotic expressions (4.3, 4.4) of the modes we obtain

$$Q_\omega^{R,L} = \theta_{R,L} \left[|C_+^\omega|^2 + \operatorname{sgn}(\omega)\, |D_-^\omega|^2 \right]$$
$$= \theta_{R,L} \left[|C_-^\omega|^2 + \operatorname{sgn}(\omega)\, |D_+^\omega|^2 \right] \qquad , \qquad (4.6)$$

in accordance with the wronskian theorem. By taking as surfaces of integration $\tau = \pm\infty$, we easily see that each term of these sums represents a fraction of the total charge of the mode that comes from past or goes to future components of the boundaries of the patches **R** and **L**. More precisely we have:

$$Q_\omega^R = Q_\omega^{\mathcal{I}_{R,F}^+} + Q^{\mathcal{H}_R^+} \qquad Q_\omega^L = Q_\omega^{\mathcal{I}_{L,F}^+} + Q^{\mathcal{H}_L^+}$$
$$= Q_\omega^{\mathcal{I}_{R,P}^-} + Q^{\mathcal{H}_R^-} \qquad = Q_\omega^{\mathcal{I}_{L,P}^-} + Q^{\mathcal{H}_L^-} \qquad , \qquad (4.7)$$

---

[4]The solution of this equation will be expressed in terms of Whittaker's functions in the next section.

[5]In principle, we would have had to indicate, for each mode, the patches on which it is considered, and index the various coefficients C and D that it defines by labels R, L, P or F. As we found the notations cumbersome enough as they are, we shall omit these precisions, hoping that the context will always be clear enough to avoid ambiguities.

with the various charges associated to the mode ϕ_ω given by the coefficients appearing in its asymptotic expansion: $Q_\omega^{\mathcal{I}_{R,F}^+} = +|C_+^\omega|^2$, $Q^{\mathcal{H}_R^+} = +\operatorname{sgn}(\omega)\,|D_-^\omega|^2$...

On the quadrant $\mathbf{P}$ and $\mathbf{F}$, the scalar product is given by the integration of the current (3.2) over $\xi = C^{te}$ surfaces. It expresses itself in terms of the Wronskian of the solutions $\mathcal{F}_\omega^*$ and $\mathcal{F}_\omega$ of eq. (4.2) as:

$$\begin{aligned}
< \phi_\omega, \phi_{\omega'} >_{P,F} &= \theta_{P,F}\, i\, \mathcal{F}_\omega^* \overset{\leftrightarrow}{\partial}_\xi \mathcal{F}_\omega \delta(\omega - \omega') \\
&\equiv Q_\omega^{P,F} \delta(\omega - \omega')
\end{aligned} \tag{4.8}$$

where, due to the time orientation of the vector field ∂_ξ we have $\theta_F = 1, \theta_P = -1$.

Here the evaluation of the charge is simpler. The Wronskian theorem tells us that:

$$\begin{aligned}
Q_\omega^{P,F} &= \theta_{P,F}\operatorname{sgn}(\omega)\left[|D_-^\omega|^2 - |D_+^\omega|^2\right] \\
&= \theta_{P,F}\left[|C_+^\omega|^2 - |C_-^\omega|^2\right]
\end{aligned} \tag{4.9}$$

As in eq. (4.7), the total charge $Q_\omega^{P,F}$ can be split in terms of the amounts of charge crossing the different components of the boundaries of the Rindler patches. We obtain :

$$\begin{aligned}
Q_\omega^F &= Q^{\mathcal{H}_L^+} + Q^{\mathcal{H}_R^+} & Q_\omega^P &= Q_\omega^{\mathcal{H}_R^-} + Q_\omega^{\mathcal{H}_L^-} \\
&= Q_\omega^{\mathcal{I}_{F,L}^+} + Q_\omega^{\mathcal{I}_{F,R}^+} & &= Q_\omega^{\mathcal{I}_{P,L}^-} + Q_\omega^{\mathcal{I}_{P,R}^-}
\end{aligned} \quad, \tag{4.10}$$

with $Q_\omega^{\mathcal{I}_{F,R}^+} = +|C_+^\omega|^2$, $Q^{\mathcal{I}_{P,R}^-} = +|C_-^\omega|^2$...

The particle assignment of these modes is obtained by considering the asymptotic behaviors of wave packets of almost fixed value of ω built on these modes, for instance slightly spread around a value $\bar\omega$ with a Gaussian weight. Indeed, asymptotically, they are expressed as superpositions of W.K.B. solutions $\exp iS(\tau, \xi, \omega)$ where $S(\tau, \xi, \omega)$ is the Maupertuis action for the Hamiltonian (2.3). By evaluating these superpositions in the saddle point approximation we see that these packets are localized on classical trajectories $\partial S/\partial\omega|_{\bar\omega} = C^{te}$. This allows to interpret them in terms of particles, in the light of the analysis of the previous section. Moreover, having this particle interpretation of the modes, we will obtain, in the subsequent subsections, a precise description of the Schwinger mechanism of pair creation in the Rindler vacua; fig. (**3**) summarizes the results.

4.1 Quantization on the right quadrant (R)

In Appendix **A** we discuss the solutions of eq. (4.1) in terms of Whittaker's functions and, using integral representations of them, we give the expression of the various coefficients needed to determine the charges carried by the modes. These considerations lead us to define on the right quadrant *in*- and *out*-modes as follows:

- Particles *in*-modes are given by the functions:

$$\mathcal{U}_{in,R}^\omega = \mathcal{N}(\mathcal{U}_{in,R}^\omega) \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} e^{-a\xi} W_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}), i\frac{\omega}{2a}}\left[-i\frac{qE}{2a^2}e^{2a\xi}\right] \tag{4.11}$$

10

whose supports (in terms of wave packet) are on $\mathcal{H}_R^-$, $\mathcal{H}_R^+$ and $\mathcal{I}_R^+$ and

$$\mathcal{V}_{in,R}^\omega = \mathcal{N}(\mathcal{V}_{in,R}^\omega)\frac{e^{-i\omega\tau}}{\sqrt{2\pi}}e^{-a\xi}M_{-i(\frac{\omega}{2a}-\frac{m^2}{2qE}),-i\frac{\omega}{2a}}\left[+i\frac{qE}{2a^2}e^{2a\xi}\right] \tag{4.12}$$

whose supports are on $\mathcal{I}_R^-, \mathcal{I}_R^+$ and $\mathcal{H}_R^+$. Obviously $\mathcal{V}_{in}^\omega$ and $\mathcal{U}_{in}^\omega$ modes are orthogonal; their supports do not overlap on the Cauchy surface $\mathcal{H}_R^- \cup \mathcal{I}_R^-$. They are also orthogonal among themselves:

$$\begin{aligned}\langle\mathcal{U}_{in,R}^\omega, \mathcal{U}_{in,R}^{\omega'}\rangle &= |\mathcal{N}(\mathcal{U}_{in,R}^\omega)|^2\left(\frac{qE}{a}\right)\frac{e^{\frac{\pi\omega}{2a}}}{\sinh(\pi\omega/a)}\cosh\left[\pi\left(\frac{\omega}{a}-\frac{m^2}{2qE}\right)\right]\delta(\omega-\omega')\\ &\equiv Q(\mathcal{U}_{in,R}^\omega)\delta(\omega-\omega') \quad ,\\ \langle\mathcal{V}_{in,R}^\omega, \mathcal{V}_{in,R}^{\omega'}\rangle &= |\mathcal{N}(\mathcal{V}_{in,R}^\omega)|^2\left(\frac{\omega/a}{\sinh(\pi\omega/a)}\right)\left(\frac{qE}{a}\right)e^{\frac{\pi}{2}(\frac{\omega}{2a}+\frac{m^2}{qE})}\cosh\left[\pi\left(\frac{\omega}{a}-\frac{m^2}{2qE}\right)\right]\delta(\omega-\omega')\\ &\equiv Q(\mathcal{V}_{in,R}^\omega)\delta(\omega-\omega') \quad . \end{aligned} \tag{4.13}$$

The modes $\mathcal{U}_{in,R}^\omega$ enter into $\mathbf{R}$ via $\mathcal{H}_R^-$ while the modes $\mathcal{V}_{in,R}^\omega$ emerge from $\mathcal{I}_R^-$. They are normalized such that $Q(\mathcal{U}_{in,R}^\omega) = \text{sgn}(\omega) = \pm 1$ and $Q(\mathcal{V}_{in,R}^\omega) = +1$ represent their total charge. From our previous discussion we obtain :

$$\begin{aligned}Q_{\mathcal{H}_R^+}(\mathcal{U}_{in,R}^\omega) &= \text{sgn}(\omega)\frac{e^{-\frac{\pi\omega}{a}}\cosh\left[\pi\frac{m^2}{2qE}\right]}{\cosh\left[\pi\left(\frac{\omega}{a}-\frac{m^2}{2qE}\right)\right]} \equiv \text{sgn}(\omega)\,q_1 \quad ,\\ Q_{\mathcal{I}_R^+}(\mathcal{U}_{in,R}^\omega) &= \frac{e^{-\frac{\pi m^2}{2qE}}\left|\sinh\left[\pi\frac{\omega}{a}\right]\right|}{\cosh\left[\pi\left(\frac{\omega}{a}-\frac{m^2}{2qE}\right)\right]} \equiv q_2 \quad . \end{aligned} \tag{4.14}$$

In the same way we obtain:

$$Q_{\mathcal{I}_R^+}(\mathcal{V}_{in,R}^\omega) = q_1 \quad , \quad Q_{\mathcal{H}_R^+}(\mathcal{V}_{in,R}^\omega) = \text{sgn}(\omega)\,q_2 \quad ,$$
$$Q(\mathcal{V}_{in,R}^\omega) = Q_{\mathcal{I}_R^-}(\mathcal{V}_{in,R}^\omega) = Q_{\mathcal{H}_R^+}(\mathcal{V}_{in,R}^\omega) + Q_{\mathcal{I}_R^+}(\mathcal{V}_{in,R}^\omega) = +1 \quad . \tag{4.15}$$

The interpretation of these equations is obvious. The modes $\mathcal{U}_{in,R}^\omega$ describe incoming particles when $\omega > 0$ and incoming anti-particles when $\omega < 0$. The modes $\mathcal{V}_{in,R}^\omega$ always describe incoming particles. A schematic drawing of wave packets built with these modes are depicted on fig. (**3**). During their voyage in $\mathbf{R}$, each incoming mode splits into two outgoing branches, one ending on $\mathcal{I}_R^+$ and the other crossing $\mathcal{H}_R^+$ when they leave the quadrant. Charge conservation implies that:

$$Q = Q_{\mathcal{H}_R^+} + Q_{\mathcal{I}_R^+} \quad .$$

Moreover, in a constant electric field only positively (resp. negatively) charged particles can arrive from $\mathcal{I}_R^-$ (resp. $\mathcal{I}_L^-$) and go up to $\mathcal{I}_R^+$ (resp. $\mathcal{I}_R^-$). This is reflected by the positivity of $Q_{\mathcal{I}_R^-}(\mathcal{V}_{in}^\omega)$ and $Q_{\mathcal{I}_R^+}(\mathcal{U}_{in,R}^\omega)$. The charged scalar quantum field operator can be represented on $\mathbf{R}$ as

$$\begin{aligned}\hat{\phi}_R = \int_{-\infty}^{+\infty}d\omega[&a_{\mathcal{V}_R}^{in}(\omega)\mathcal{V}_{in,R}^\omega + \theta(\omega)a_{\mathcal{U}_R}^{in}(\omega)\mathcal{U}_{in,R}^\omega\\ &+\theta(-\omega)b_{\mathcal{U}_R}^{+\,in}(\omega)\mathcal{U}_{in,R}^\omega] \quad , \end{aligned} \tag{4.16}$$

11

the various field amplitude operators being defined as usual by:

$$a_{\mathcal{U}}^{in}(\omega)|0\rangle_{\mathcal{U}_{in}^\omega}^{\mathbf{R}} = 0 \quad \omega > 0 \quad , \quad b_{\mathcal{U}}^{in}(\omega)|0\rangle_{\mathcal{U}_{in}^\omega}^{\mathbf{R}} = 0 \quad \omega < 0 \quad , \quad a_{\mathcal{V}}^{in}(\omega)|0\rangle_{\mathcal{V}_{in}^\omega}^{\mathbf{R}} = 0 \quad ,$$

and the $\mathbf{R}$ *in*-vacuum state[6] as the tensorial product

$$|0\rangle_{in}^{\mathbf{R}} = \prod_{\omega=-\infty}^{+\infty} \left(|0\rangle_{\mathcal{U}_{in}^\omega}^{\mathbf{R}} \otimes |0\rangle_{\mathcal{V}_{in}^\omega}^{\mathbf{R}} \right). \tag{4.17}$$

• Similarly we may consider modes associated to outgoing particles. They are given by:

$$\mathcal{V}_{out,R}^\omega = \mathcal{N}(\mathcal{V}_{out,R}^\omega) \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} e^{-a\,\xi} W_{-i(\frac{\omega}{2a} - \frac{m^2}{2qE}), i\frac{\omega}{2a}} \left[+i\frac{qE}{2a} e^{2a\,\xi} \right] \quad , \tag{4.18}$$

with $|\mathcal{N}(\mathcal{V}_{out,R}^\omega)| = |\mathcal{N}(\mathcal{U}_{in,R}^\omega)|$. They correspond to modes entering into $\mathbf{R}$ from $\mathcal{H}_R^-$ and $\mathcal{I}_R^-$ and leaving it via $\mathcal{H}_R^+$. They carry the asymptotic charges

$$Q(\mathcal{V}_{out,R}^\omega) = Q_{\mathcal{H}_R^+}(\mathcal{V}_{out,R}^\omega) = \mathrm{sgn}(\omega) \quad , \quad Q_{\mathcal{H}_R^-}(\mathcal{V}_{out,R}^\omega) = \mathrm{sgn}(\omega)\, q_1 \quad , \quad Q_{\mathcal{I}_R^-}(\mathcal{V}_{out,R}^\omega) = q_2 \quad . \tag{4.19}$$

These modes are also orthogonal. They describe outgoing particles ($\omega > 0$) or anti-particles ($\omega < 0$) and of course obey the charge conservation rule:

$$Q(\mathcal{V}_{out,R}^\omega) = Q_{\mathcal{H}_R^-}(\mathcal{V}_{out,R}^\omega) + Q_{\mathcal{I}_R^-}(\mathcal{V}_{out,R}^\omega) = \mathrm{sgn}(\omega) \tag{4.20}$$

with $Q_{\mathcal{I}_R^-}(\mathcal{V}_{out,R}^\omega)$ always positive. As for the *in*-modes, to obtain a complete set on $\mathbf{R}$, we have to add to the set of functions (4.18) the modes

$$\mathcal{U}_{out,R}^\omega = \mathcal{N}(\mathcal{U}_{out,R}^\omega) \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} e^{-a\xi} M_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}), i\frac{\omega}{2a}} \left[-i\frac{qE}{2a^2} e^{2a\,\xi} \right] \quad , \tag{4.21}$$

which are also orthogonal among themselves and with respect to the set $\{\mathcal{V}_{out,R}^\omega\}$. These modes come from $\mathcal{H}_R^-$ and $\mathcal{I}_R^-$ and end on $\mathcal{I}_R^+$ and $\mathcal{I}_R^+$. They charge content is

$$Q_{\mathcal{I}_R^+}(\mathcal{U}_{out,R}^\omega) = 1 \quad , \quad Q_{\mathcal{I}_R^-}(\mathcal{U}_{out,R}^\omega) = q_1 \quad , \quad Q_{\mathcal{H}_R^-}(\mathcal{U}_{out,R}^\omega) = \mathrm{sgn}(\omega)\, q_2 \quad . \tag{4.22}$$

These charge are such that:

$$Q(\mathcal{U}_{out}^\omega) = Q_{\mathcal{H}_R^-}(\mathcal{U}_{out}^\omega) + Q_{\mathcal{I}_R^-}(\mathcal{U}_{out}^\omega) = 1. \tag{4.23}$$

With these modes we may express the quantum field operator (4.16) as

$$\hat{\phi}_R = \int_{-\infty}^{+\infty} d\omega [a_{\mathcal{U}_R}^{out}(\omega)\mathcal{U}_{out,R}^\omega + \theta(\omega)a_{\mathcal{V}_R}^{out}(\omega)\mathcal{V}_{out,R}^\omega$$
$$+ \theta(-\omega)b_{\mathcal{V}_R}^{+\,out}(\omega)\mathcal{V}_{out,R}^\omega] \quad . \tag{4.24}$$

---

[6]This vacuum state is not the end of the story. Indeed the true vacuum state must be defined on a Cauchy surface of the whole space, i.e. it must include a factor representing the $\mathbf{L}$ part of the vacuum.

Of course *out-* and *in*-modes are related by a Bogoljubov transformation. Using formulas recalled in Appendix **A**, we easily obtain :

$$\mathcal{U}_{out,R}^{\omega} = \theta(\omega)\alpha_{\mathcal{U}\mathcal{U}}^{R}(\omega)\mathcal{U}_{in,R}^{\omega} + \alpha_{\mathcal{U}\mathcal{V}}^{R}(\omega)\mathcal{V}_{in,R}^{\omega} + \theta(-\omega)\beta_{\mathcal{U}\mathcal{U}}^{R}(\omega)\mathcal{U}_{in,R}^{\omega} \qquad , \tag{4.25}$$

$$\mathcal{V}_{out,R}^{\omega} = \theta(\omega)\left\{\alpha_{\mathcal{V}\mathcal{U}}^{R}(\omega)\mathcal{U}_{in,R}^{\omega} + \alpha_{\mathcal{V}\mathcal{V}}^{R}(\omega)\mathcal{V}_{in,R}^{\omega}\right\} + \theta(-\omega)\left\{\gamma_{\mathcal{V}\mathcal{V}}^{R}(\omega)\mathcal{V}_{in,R}^{\omega} + \epsilon_{\mathcal{V}\mathcal{U}}^{R}(\omega)\mathcal{U}_{in,R}^{\omega}\right\} \tag{4.26}$$

whose coefficients, fully displayed in Appendix **B**, are such that:

$$|\alpha_{\mathcal{U}\mathcal{V}}^{R}(\omega)|^2 = |\epsilon_{\mathcal{V}\mathcal{U}}^{R}(\omega)|^2 = |\alpha_{\mathcal{V}\mathcal{U}}^{R}(\omega)|^2 = q_1 \qquad ,$$
$$|\alpha_{\mathcal{U}\mathcal{U}}^{R}(\omega)|^2 = |\beta_{\mathcal{U}\mathcal{U}}^{R}(\omega)|^2 = |\gamma_{\mathcal{V}\mathcal{V}}^{R}(\omega)|^2 = q_2 \qquad , \tag{4.27}$$

and (remember that q_1 and q_2 are function of ω):

$$|\alpha_{\mathcal{U}\mathcal{V}}^{R}(\omega < 0)|^2 - |\beta_{\mathcal{U}\mathcal{U}}^{R}(\omega < 0)|^2 \equiv q_1 - q_2 = 1 \qquad ,$$
$$|\alpha_{\mathcal{U}\mathcal{V}}^{R}(\omega > 0)|^2 + |\alpha_{\mathcal{U}\mathcal{U}}^{R}(\omega > 0)|^2 \equiv q_1 + q_2 = 1 \qquad . \tag{4.28}$$

The last relation shows that the Bogoljubov transformation between the *in-* and *out-* vacuum state factors is non trivial only for $\omega < 0$, in which case we obtain

$$|0\rangle_{\mathcal{U}_{in}^{\omega}}^{\mathbf{R}}|0\rangle_{\mathcal{V}_{in}^{\omega}}^{\mathbf{R}} = |\alpha_{\mathcal{U}\mathcal{V}}(\omega)|^{-1}e^{-\frac{\gamma_{\mathcal{V}\mathcal{V}}(\omega)}{\alpha_{\mathcal{U}\mathcal{V}}(\omega)}a_{\mathcal{U}}^{\dagger\,out}(\omega)b_{\mathcal{V}}^{\dagger\,out}(\omega)}|0\rangle_{\mathcal{U}_{out}^{\omega}}^{\mathbf{R}}|0\rangle_{\mathcal{V}_{out}^{\omega}}^{\mathbf{R}} \qquad (\forall\omega < 0), \tag{4.29}$$

reflecting the instability of the *in*-vacuum and the pair production predicted by the Schwinger mechanism. The fact that only the $\omega < 0$ vacuum factors appears to be unstable is in accord with formulas (4.14),(4.15) which indicate that it is only for these values of ω that an *in*-mode of a given charge generate *out*-branches with opposite charges. Note also that this instability can be interpreted as a tunneling between classical trajectory living in **R** and its "partner" when this latter visits the **R** quadrant.

The probability of vacuum persistence is given by:

$$\begin{aligned}
|^{\mathbf{R}}\!< 0, in|0, out >^{\mathbf{R}}|^2 &= \prod_{\omega<0}|\alpha_{\mathcal{U}\mathcal{V}}(\omega)|^{-2} \\
&= \exp\left[-\sum_{\omega<0}\ln\left(1 + |\beta_{\mathcal{U}\mathcal{U}}^{R}|^2\right)\right] \\
&= \exp\left[-\sum_{\omega<0}\ln\left(\frac{1 + e^{-\pi\frac{m^2}{qE}}}{1 + e^{-\pi\frac{m^2}{qE}}e^{2\frac{\pi\omega}{a}}}\right)\right] \qquad . \tag{4.30}
\end{aligned}$$

The significance of a symbol as $\sum_{\omega<0}$ is analyzed in ref. [9, 8] and recalled at the end of section **3**. The density of modes of frequency ω, in an interval of width $\mathcal{T}$ in the τ variable, is $\mathcal{T}/2\pi\,d\omega$. The sum over $\omega < 0$ corresponds to consider essentially the particle trajectories whose mirror trajectories also visit the **R** quadrant. The spacelike coordinate ξ_c of the center of contributing trajectories are obtained immediately from equation (2.9) which gives $d\omega = \frac{am^2}{qE}d\Omega = -aqE\rho_c d\rho_c$ and thus $\mathcal{T}\,d|\omega| = qE\,dV$. As a consequence we obtain:

$$|^{\mathbf{R}}\!< 0, in|0, out >^{\mathbf{R}}|^2 = \exp\left[-\ln\left(1 + e^{-\pi\frac{m^2}{qE}}\right)\frac{qE}{2\pi}V + \mathcal{T}\times\text{finite terms}\right] \qquad , \tag{4.31}$$

13

where in the limit of large space-time volume **V** (i.e. large interval $[-\omega_1, -\omega_2]$ of length $\Delta\omega \gg a$), we have

$$\mathbf{V} = \frac{\mathcal{T}}{qE}\Delta\omega = \frac{\mathcal{T}}{2\,a}\left(\exp[2a\xi_c(-\omega_2)] - \exp[2a\xi_c(-\omega_1)]\right) \equiv T\ R \quad , \tag{4.32}$$

where we have denoted the difference of Rindler radii by $R = \rho_c(\omega_2) - \rho_{c1}(\omega_1)$ and introduced a mean proper time $T = \mathcal{T}(\rho_c[\omega_2] + \rho_{c1}(\omega_1))/(2\,a) \equiv \mathcal{T}\,\bar{R}$. ¿From eq. (4.31) we see that the rate of Rindler pair creation from Rindler right quadrant vacuum consists in two pieces. The first one, as expected, is proportional to the measure of the volume multiplied by the Schwinger rate. To this dominant (in the large volume limit) term, a surface correction, which can be expressed in term of dilogarithm (Spence) function Li_2, has to be subtracted:

$$\sum_{\omega<0} 1 + e^{-\pi\frac{m^2}{qE}}e^{2\frac{\pi\omega}{a}} = \frac{\mathcal{T}}{2\,\pi}\int_{-\omega_2}^{-\omega_1}\left(1 + e^{-\pi\frac{m^2}{qE}}e^{-2\frac{\pi\omega}{a}}\right)d\omega$$

$$= \frac{a\,\mathcal{T}}{4\,\pi^2}\left[Li_2(-e^{-\pi\,qE\,\rho_{c2}^2}) - Li_2(-e^{-\pi\,qE\,\rho_{c1}^2})\right] \quad .$$

As emphasized by Robert Brout [10], it is instructive to compare the formula for the persistence of Rindler vacuum, eq. (4.30), with a similar situation which arises in the theory of Hawking black hole radiation. In particular, our result stands in strong analogy to the analysis of s-wave emission by a Schwarzschild black hole, and even more so when this latter is reduced to a pseudo $(1+1)$ dimensional problem, when one neglects the effects of the s-wave ''centrifugal" barrier (See ref. [11]). It will be noted that the rate eq. (4.30) defines is a difference between a volume term, proportional to R, and a surface term wherein R is replaced by a characteristic length $a/(qE\,\bar{R})$. Each of these terms is proportional to T, hence giving rise to a rate per unit volume and rate per unit surface respectively. For the Schwinger case, the object of our present study, R is macroscopic and the surface term is negligible. But for the case of s-wave black hole evaporation, the steady state production comes from a region of $\mathcal{O}(M)$ of the horizon where $2\,M$ is the Schwarzschild radius. Recall the formula for the rate of particle emission

$$\frac{\langle T_o^r\rangle}{k_B\,T_{\text{Hawk.}}} = \frac{\pi}{12}k_B T_{\text{Hawk.}} \quad . \tag{4.33}$$

There is a prefactor M in the above, the volume of the "skin"; this replaces R in our formula.

Equation (4.33) contains the vacuum expectation value of the energy flux in Unruh vacuum, the outgoing mode piece of the Hartle-Hawking vacuum. In ref. [11] there is displayed the same calculation in Boulware vacuum, the analog of what we have called Rindler vacuum. It is shown that in this case vacuum polarization effects cancel against the emission effects and $\langle T_o^r\rangle_{\text{Boulware}} = 0$. In our eq. (4.31), this subtraction comes up in analogous fashion, due to the particular structure of the Rindler modes, and hence of Rindler vacuum, in the vicinity of the horizons. The difference between the two cases is that in the black hole case the cancellation of $\langle T_{or}\rangle_{Boulware}$ must be total. This is a consequence of the static character of the Schwarzschild metric which underlies the construction of Boulware vacuum. In the Schwinger case the problem is actually intrinsically time dependent. Indeed the electric field has to be (adiabatically) switched on and off in order to prepare the *in*-vacuum initial state (or to observe the *out*-one).

14

4.2 Quantization on the left quadrant (L)

Formally the field equations and their solutions are the same on the left and right quadrants. Their only difference results that on **L** the vector field ∂_τ points near the past. This implies that the signs of the charges and the *in* and *out* labels have to be interchange with respect to their values on the **R** quadrant. The analytical expression of the *in-* and *out-*modes on the **L**-quadrant can be deduced immediately from the drawing (see fig. (**3**)) of the supports of the modes built on the **R**-quadrant. Adding a extra subscript $(R, L, \text{etc} \dots)$ to make the distinction between the modes defined on the various quadrants, we obtain for the *in*-mode the expression:

$$
\begin{aligned}
\mathcal{U}_{in,L}^\omega(\xi_L, \tau_L) &= \mathcal{N}(\mathcal{U}_{in,L}^\omega) \frac{e^{-i\omega\tau_L}}{\sqrt{2\pi}} e^{-a\,\xi_L} M_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}), i\frac{\omega}{2a}} \left[-i\frac{qE}{2a^2} e^{2a\xi_L} \right] \\
&\propto \mathcal{U}_{out,R}^\omega(\xi_L, \tau_L)
\end{aligned}
\tag{4.34}
$$

and for their charge content:

$$
\begin{aligned}
Q(\mathcal{U}_{in,L}^\omega) &= Q_{\mathcal{I}_L^-}(\mathcal{U}_{in,L}^\omega) = -1 = -Q_{\mathcal{I}_R^+}(\mathcal{U}_{out,R}^\omega) &, \\
Q_{\mathcal{H}_L^+}(\mathcal{U}_{in,L}^\omega) &= -Q_{\mathcal{H}_R^-}(\mathcal{U}_{out,R}^\omega) = -\operatorname{sgn}(\omega)\, q_2 &, \\
Q_{\mathcal{I}_L^+}(\mathcal{U}_{in,L}^\omega) &= -Q_{\mathcal{I}_L^-}(\mathcal{U}_{out,R}^\omega) = -q_1 &.
\end{aligned}
\tag{4.35}
$$

Similarly

$$
\mathcal{V}_{in,L}^\omega(\xi_L, \tau_L) \propto \mathcal{V}_{out,R}^\omega(\xi_L, \tau_L) \quad,
\tag{4.36}
$$

$$
\mathcal{U}_{out,L}^\omega(\xi_L, \tau_L) \propto \mathcal{U}_{in,R}^\omega(\xi_L, \tau_L) \quad,
\tag{4.37}
$$

$$
\mathcal{V}_{out,L}^\omega(\xi_L, \tau_L) \propto \mathcal{V}_{in,R}^\omega(\xi_L, \tau_L) \quad.
$$

Here again, we may define *in-* and *out-* vacua, Fock spaces, etc ... and the amplitude of vacuum persistence is given by an expression similar to (4.31).

4.3 Quantization on the past quadrant (P)

On the past quadrant, it is the vector field $-\partial_{\xi_p}$ that defines the future direction. The various modes can again be expressed again in terms of Whittaker's functions: with their charges given by eqs (4.8). More precisely we may choose as particle incoming modes

$$
\mathcal{V}_{in,P}^\omega(\xi_P, \tau_P) = \mathcal{N}(\mathcal{V}_{in,P}^\omega) \frac{e^{-i\omega\tau_P}}{\sqrt{2\pi}} e^{-a\xi_P} W_{-i(\frac{\omega}{2a} - \frac{m^2}{2qE}),\, i\frac{\omega}{2a}} \left[-i\frac{qE}{2a^2} e^{2a\,\xi_P} \right] \quad.
\tag{4.38}
$$

They came out from $\mathcal{I}_R^-$, carrying a charge

$$
Q_{\mathcal{I}_R^-}(\mathcal{V}_{in,P}^\omega) = |\mathcal{N}(\mathcal{V}_{in,P}^\omega)|^2 \left(\frac{qE}{a} \right) e^{-\frac{\pi}{2}(\frac{\omega}{a} - \frac{m^2}{qE})} = +1
\tag{4.39}
$$

and leave **P** via $\mathcal{H}_R^-$ and $\mathcal{H}_L^-$, taking away the charges

$$
Q_{\mathcal{H}_R^-}(\mathcal{V}_{in,P}^\omega) = e^{\pi\left(\frac{\omega}{a} - \frac{m^2}{2qE}\right)} \frac{\cosh\left[\pi\frac{m^2}{2qE}\right]}{\sinh\left[\frac{\pi\omega}{a}\right]} \equiv \operatorname{sgn}(\omega)\, q_3 \quad,
\tag{4.40}
$$

and

$$Q_{\mathcal{H}_L^-}(\mathcal{V}_{in,P}^\omega) = -\mathrm{e}^{-\frac{\pi m^2}{2qE}}\frac{\cosh\left[\pi\left(\frac{\omega}{a}-\frac{m^2}{2qE}\right)\right]}{\sinh\left[\frac{\pi\omega}{a}\right]} \equiv -\operatorname{sgn}(\omega)\,q_4 \qquad . \tag{4.41}$$

The complex conjugates of the anti-particle incoming modes are given by

$$\mathcal{U}_{in,P}^\omega(\xi_P,\tau_P) = \mathcal{N}(\mathcal{U}_{in,P}^\omega)\frac{e^{-i\omega\tau_P}}{\sqrt{2\pi}}e^{-a\,\xi_P}W_{i(\frac{\omega}{2a}-\frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[i\frac{qE}{2a^2}e^{2a\,\xi_P}\right] \tag{4.42}$$

$$\propto \left[\mathcal{V}_{in,P}^\omega(\xi_P,-\tau_P)\right]^* \qquad . \tag{4.43}$$

Their asymptotic charge content, when they appears on $\mathcal{I}_L^-$, is:

$$Q_{\mathcal{I}_L^-}(\mathcal{U}_{in,P}^\omega) = -1 \qquad . \tag{4.44}$$

This charge splits into

$$\begin{aligned}
Q_{\mathcal{H}_R^-}(\mathcal{U}_{in,P}^\omega) &= \operatorname{sgn}(\omega)\,q_4 \qquad , \\
Q_{\mathcal{H}_L^-}(\mathcal{U}_{in,P}^\omega) &= -\operatorname{sgn}(\omega)\,q_3 \qquad ,
\end{aligned} \tag{4.45}$$

with $\operatorname{sgn}(\omega)\,(q_3 - q_4) = 1$.
Outgoing $\mathcal{U}$-modes on $\mathbf{P}$ have to vanish on $\mathcal{H}_L^-$ i.e. their coefficient D_- must be zero. They are given by Whittaker's M-functions:

$$\mathcal{U}_{out,P}^\omega(\xi_P,\tau_P) = \mathcal{N}(\mathcal{U}_{out,P}^\omega)\frac{e^{-i\omega\tau_P}}{\sqrt{2\pi}}e^{-a\xi_P}M_{i(\frac{\omega}{2a}-\frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[i\frac{qE}{2a^2}e^{2a\,\xi_P}\right] \tag{4.46}$$

with

$$|\mathcal{N}(\mathcal{U}_{out,P}^\omega)|^{-2} = |D_+(\mathcal{M}_-^+)|^2 = |\omega|\left(\frac{qE}{2a^2}\right)\mathrm{e}^{-\frac{\pi\omega}{2a}} \tag{4.47}$$

whose charge content is

$$\begin{aligned}
Q(\mathcal{U}_{out,P}^\omega) &\equiv Q_{\mathcal{H}_R^-}(\mathcal{U}_{out,P}^\omega) = +\operatorname{sgn}(\omega) \qquad , \\
Q_{\mathcal{I}_L^-}(\mathcal{U}_{out,P}^\omega) &= -q_4 \qquad < 0 \qquad , \\
Q_{\mathcal{I}_R^-}(\mathcal{U}_{out,P}^\omega) &= q_3 \qquad > 0 \qquad .
\end{aligned} \tag{4.48}$$

Accordingly, modes with positive (resp. negative) values of ω have to be associated to particles (resp. antiparticles) and, as expected, the sum of the two last charges reproduces the total charge of the mode. Finally, we have also to consider the modes

$$\mathcal{V}_{out,p}^\omega(\xi_P,\tau_P) = \mathcal{N}(\mathcal{V}_{out,P}^\omega)\frac{e^{-i\omega\tau_P}}{\sqrt{2\pi}}e^{-a\xi_P}M_{-i(\frac{\omega}{2a}-\frac{m^2}{2qE}),-i\frac{\omega}{2a}}\left[-i\frac{qE}{2a^2}e^{2a\,\xi_P}\right] \tag{4.49}$$

$$\propto \left[\mathcal{U}_{in,P}^\omega(\xi_P,-\tau_P)\right]^* \qquad . \tag{4.50}$$

that emerges from $\mathcal{I}^-$ and leaves $\mathbf{P}$ via $\mathcal{H}_R^-$, with the charges

$$\begin{aligned}
Q(\mathcal{V}_{out,P}^\omega) &\equiv Q_{\mathcal{H}_L^-}(\mathcal{V}_{out,P}^\omega) = -\operatorname{sgn}(\omega) \qquad , \\
Q_{\mathcal{I}_L^-}(\mathcal{V}_{out,P}^\omega) &= -q_3 \qquad , \\
Q_{\mathcal{I}_R^-}(\mathcal{V}_{out,P}^\omega) &= q_4 \qquad .
\end{aligned} \tag{4.51}$$

Here also the quantum field operator can be expressed in two equivalent forms, using the *in-* and *out-* modes:

$$\hat{\phi}_P = \int_{-\infty}^{+\infty} d\omega \left[a_{\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{in,P}^{\omega} + b_{\mathcal{U}_P}^{+\ in}(\omega)\mathcal{U}_{in,P}^{\omega} \right] \tag{4.52}$$

$$= \int_{-\infty}^{+\infty} d\omega \left[\theta(\omega) \left(a_{\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{out,P}^{\omega} + b_{\mathcal{V}_P}^{+\ out}(\omega)\mathcal{V}_{out,P}^{\omega} \right) \right.$$
$$\left. + \theta(-\omega) \left(a_{\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{out,P}^{\omega} + b_{\mathcal{U}_P}^{+\ out}(\omega)\mathcal{U}_{out,P}^{\omega} \right) \right] \qquad . \tag{4.53}$$

Using the relations given in Appendix **A**, we obtain easily the Bogoljubov transformation relating *in-* and *out*-modes:

$$\mathcal{U}_{out,P}^{\omega} = \theta(\omega) \left\{ \alpha_{\mathcal{U}\mathcal{V}}^P(\omega)\mathcal{V}_{in,P}^{\omega} + \beta_{\mathcal{U}\mathcal{U}}^P(\omega)\mathcal{U}_{in,P}^{\omega} \right\} + \theta(-\omega) \left\{ \gamma_{\mathcal{U}\mathcal{V}}^P(\omega)\mathcal{V}_{in,P}^{\omega} + \epsilon_{\mathcal{U}\mathcal{U}}^P(\omega)\mathcal{U}_{in,P}^{\omega} \right\} \quad ,$$

$$\mathcal{V}_{out,P}^{\omega} = \theta(-\omega) \left\{ \alpha_{\mathcal{V}\mathcal{V}}^P(\omega)\mathcal{V}_{in,P}^{\omega} + \beta_{\mathcal{V}\mathcal{U}}^P(\omega)\mathcal{U}_{in,P}^{\omega} \right\} + \theta(\omega) \left\{ \gamma_{\mathcal{V}\mathcal{V}}^P(\omega)\mathcal{V}_{in,P}^{\omega} + \epsilon_{\mathcal{V}\mathcal{U}}^P(\omega)\mathcal{U}_{in,P}^{\omega} \right\} \tag{4.54}$$

whose coefficients (see Appendix **B**) satisfy charge conservation relations:

$$|\alpha_{\mathcal{U}\mathcal{V}}^P(\omega > 0)|^2 - |\beta_{\mathcal{U}\mathcal{U}}^P(\omega > 0)|^2 = 1 \quad , \tag{4.55}$$

and:

$$|\gamma_{\mathcal{U}\mathcal{V}}^P(\omega < 0)|^2 - |\epsilon_{\mathcal{U}\mathcal{U}}^P(\omega < 0)|^2 = -1 \quad . \tag{4.56}$$

From these equations we may repeat the steps leading to the probability (4.31) of no pair creation and obtain:

$$|^{\mathbf{P}}\!< 0, in|0, out >^{\mathbf{P}}|^2 = \exp\left\{ -\left[\sum_{\omega > 0} \ln\left(1 + |\beta_{\mathcal{U}\mathcal{U}}^P|^2\right) + \sum_{\omega < 0} \ln\left(1 + |\gamma_{\mathcal{U}\mathcal{V}}^P|^2\right) \right] \right\}$$

$$= \exp\left[-\sum_{\omega > 0} \ln\left(1 + e^{-\pi\frac{m^2}{2qE}} \frac{\cosh \pi(\frac{\omega}{a} - \frac{m^2}{2qE})}{\sinh\left[\frac{\pi\omega}{a}\right]} \right) \right.$$

$$\left. \left(1 + e^{-\pi\left(\frac{\omega}{a} + \frac{m^2}{2qE}\right)} \frac{\cosh\left[\pi\frac{m^2}{2qE}\right]}{\sinh\left[\frac{\pi\omega}{a}\right]} \right) \right] \qquad . \tag{4.57}$$

4.4 Quantization on the future quadrant (F)

The modes on the **F** quadrant are obtained from those defined on **P** in the same way we pass from **R** modes to **L** modes :

$$\mathcal{V}_{in,F}^{\omega}(\xi_F, \tau_F) \propto \mathcal{V}_{out,P}^{\omega}(\xi_F, \tau_F) \quad ,$$
$$\mathcal{U}_{in,F}^{\omega}(\xi_F, \tau_F) \propto \mathcal{U}_{out,P}^{\omega}(\xi_F, \tau_F) \quad ,$$
$$\mathcal{V}_{out,F}^{\omega}(\xi_F, \tau_F) \propto \mathcal{V}_{in,P}^{\omega}(\xi_F, \tau_F) \quad ,$$
$$\mathcal{U}_{out,F}^{\omega}(\xi_F, \tau_F) \propto \mathcal{U}_{in,P}^{\omega}(\xi_F, \tau_F) \quad , \tag{4.58}$$

the proportionality factors reflecting the arbitrary phases appearing in the normalization factors. Note also that, as expected, these modes satisfy charge conjugation relations

$$\mathcal{U}_{in,F}^{\omega}(\xi_F, \tau_F) \propto \left[\mathcal{U}_{in,F}^{\omega}(\xi_F, -\tau_F) \right]^* \quad , \quad \mathcal{U}_{out,F}^{\omega}(\xi_F, \tau_F) \propto \left[\mathcal{V}_{out,F}^{\omega}(\xi_F, -\tau_F) \right]^* \quad . \tag{4.59}$$

Figure **3** summarizes all the discussion of this section. We have schematically represented, for the four Rindler quadrants, the behavior of typical wavepackets built with the different modes discussed here above and the charge content that each of them carries when it crosses the horizon or reaches infinity.

5 Vacuum decay rates revisited

In this section we shall reobtain the rates of vacuum decay by using functional methods. Our motivation is that we hope in this way to obtain an alternative picture of the physics underlying the pair creation. In addition, as such methods often necessitate the use of approximations, it is useful to use them as a test in examples for which the exact answer is known since there sometimes arise physical situations in which they are the only tool available. First we shall briefly recall the general formalism and illustrate it in the context of our scalar field interacting with a constant electric field, the calculation being performed with respect to an inertial t, z coordinate system, i.e. with respect to the vacuum states defined by inertial observers. Here, the interaction being quadratic, all the integrations are gaussian and the calculations can be performed analytically. Then we shall consider the same problem with respect to accelerated observers, living in the **R** quadrant. We shall first perform our calculations using the expression of the Feynman propagator as sum of modes and recover the results obtained in the previous sections. Then we shall consider the Schwinger propagator which expres itself as a kernel integrated with respect to a time variable (Schwinger proper time[7]). Thanks to the invariance with respect to translations both in time t and τ (boost) we shall be able to reduce the expression of the Schwinger propagator to the evaluation of path integrals of ordinary one-dimensional quantum mechanical problems. We shall then evaluate these path integrals at one-loop approximation (which is exact for quadratic potentials) and compare the results to those previously obtained. Finally we shall also compute the *in-out* vacuum amplitude, using an expression for the Feynman propagator obtained by unwinding the inertial propagator, and show that it does coincide with the amplitude calculated in the secure framework of the mode analysis.

The previous vacuum persistence amplitudes obtained from "mode" calculations, may also be derived from the Green functions of the quantized field. Indeed, as is well known (see for instance the book [11]), the vacuum persistence amplitude can be expressed as a functional integral :

$$\langle 0, out|0, in\rangle_{J=J^*=0} = \mathcal{Z}[0,0] \equiv e^{i\mathcal{W}} \tag{5.1}$$

where $\mathcal{Z}[J, J^*]$ is defined as :

$$\mathcal{Z}[J, J^*] = \int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left\{iS[\phi, \phi^*] + i\int J^*\phi d^dx + i\int J\phi^* d^dx\right\} \tag{5.2}$$

in terms of the action $S[\phi, \phi^*]$ of the charged scalar field, minimally coupled to the electric field:

$$S[\phi, \phi^*] = -\frac{1}{2}\int d^dx \left\{(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)^* + (\mathcal{D}_\mu\phi)^*(\mathcal{D}^\mu\phi) + (m^2 - i\epsilon)\phi\phi^* + (m^2 + i\epsilon)\phi^*\phi)\right\}$$

---

[7]Also called fifth time in the framework of quantum field theory in 3+1 dimension.

$$= -\int dx dx' \phi^*(x)\mathcal{H}_{xx'}\phi(x') \qquad . \tag{5.3}$$

(Here, and in the following, we denote collectively by x, x', the coordinate variables of the field; the context indicating if they are Rindlerian or minkowskian.)

A standard computation allows to give a sense to the expression of $\mathcal{W}$:

$$\mathcal{W} = -i\mathrm{tr}\ln(-\mathcal{G}) = \int_{m^2}^{\infty} dm^2 \int d^d x\, G_F(x,x) \tag{5.4}$$

where $\mathcal{G} = -\mathcal{H}^{-1}$ is related to the Feynman propagator

$$G_F(x,y) = -i\langle 0, out|\mathcal{T}(\phi(x)\phi^\dagger(y))|0, in\rangle / \langle 0, out|0, in\rangle \tag{5.5}$$

by:

$$\mathcal{G}(x,y) = \frac{1}{2}\{G_F(x,y) + G_F^*(y,x)\} \qquad . \tag{5.6}$$

Thus, the imaginary part (the one which encodes the vacuum instability) of $\mathcal{W}$ reads:

$$\mathcal{I}m\mathcal{W} = \mathcal{R}e\int_{m^2}^{\infty} dm^2 \int d^d x\, \mathcal{G}(x,x) = \mathcal{R}e\int_{m^2}^{\infty} dm^2 \int d^d x\, G_F(x,x) \qquad . \tag{5.7}$$

Our purpose now is to evaluate this expression for an inertial observer and an accelerated one, using different schemes of calculation.

5.1 Mode representation of the Feynman propagator

If the asymptotic vacua are those of an inertial observer, using (t,z) coordinates, it is straightforward to obtain the Feynman propagator [12] as superposition of modes (eqs 3.9 and followings in the text) :

$$\begin{aligned} iG_F(x,x') &= \frac{\delta}{\gamma}\int d\sigma\, \varphi_\sigma^{p\,out}(x)\varphi_\sigma^{a\,out}(x') \\ &+ \theta(t-t')\int d\sigma \varphi_\sigma^{p\,out}(x)\varphi_\sigma^{p\,out\,*}(x') + \theta(t'-t)\int d\sigma \varphi_\sigma^{a\,out\,*}(x)\varphi_\sigma^{a\,out}(x') \end{aligned} \tag{5.8}$$

Therefore we get :

$$\begin{aligned} \mathcal{G}(x,x') &= -\frac{i}{2}\frac{\delta}{\gamma}\int d\sigma\left(\varphi_\sigma^{p\,out}(x)\varphi_\sigma^{a\,out}(x') - \varphi_\sigma^{p\,out\,*}(x')\varphi_\sigma^{a\,out\,*}(x)\right) \\ &- \frac{i\epsilon(t-t')}{2}\int d\sigma\left(\varphi_\sigma^{p\,out}(x)\varphi_\sigma^{p\,out\,*}(x') - \varphi_\sigma^{a\,out\,*}(x)\varphi_\sigma^{a\,out}(x')\right) \qquad . \end{aligned} \tag{5.9}$$

The first term in this last expression is traceless, so we obtain :

$$\mathcal{R}e\ \mathrm{tr}\ \mathcal{G} = \mathcal{I}m\frac{\delta}{\gamma}\int d^2x \int d\sigma\, \varphi_\sigma^{p\,out}(x)\varphi_\sigma^{a\,out}(x) \qquad . \tag{5.10}$$

19

• Using the integral representation *à la Schwinger* for the product of wave functions derived in the Appendix **C**, we can rewrite this as:

$$\mathcal{R}e \ \mathrm{tr} \ \mathcal{G} = -\mathcal{I}m \frac{e^{-\frac{\pi m^2}{qE}}}{2\pi} \frac{\sqrt{qE}}{2i\pi} \int d^2x \int d\sigma \int_0^\infty \frac{ds}{(\sinh 2qEs)^{\frac{1}{2}}} e^{-im^2 s} e^{iqE(z+\frac{\sigma}{qE})^2 \tanh qEs} \qquad (5.11)$$

and the imaginary part of the $\mathcal{W}$ amplitude reads:

$$\mathcal{I}m\mathcal{W} = \mathcal{I}m \frac{1}{2\pi} \frac{\sqrt{qE}}{2i\pi} \int d^2x \int d\sigma \int_0^\infty \frac{ds}{(s-i\frac{\pi}{qE})} \frac{e^{-im^2 s}}{(\sinh 2qEs)^{\frac{1}{2}}} e^{iqE(z+\frac{\sigma}{qE})^2 \tanh qEs} \qquad . \qquad (5.12)$$

After the t and z integrations we get, in accord with eq.(3.13):

$$\mathcal{I}m\mathcal{W} \ = \ -\frac{T}{8\pi} \int d\sigma \int_{-\infty-i\pi}^{\infty-i\pi} \frac{d\theta}{\theta \sinh\theta} e^{-i\frac{m^2}{qE}\theta} = \frac{T}{4\pi} \ln(1+e^{-\pi\frac{m^2}{qE}}) \qquad . \qquad (5.13)$$

• In Rindler coordinates $(\tau, \rho) \equiv (\tau, a^{-1}e^{a\xi})$ on **R**, the *in-out* Feynman propagator, expressed as a mode superposition, reads as :

$$
\begin{aligned}
iG_F^R(x,x') \ &= \ \theta(\tau-\tau')\left\{ \int_0^\infty d\omega \left\{ \mathcal{U}_\omega^{out}(x)\mathcal{U}_\omega^{out\,*}(x') + \mathcal{V}_\omega^{out}(x)\mathcal{V}_\omega^{out\,*}(x') \right\} + \right. \\
&\qquad \left. \int_{-\infty}^0 d\omega \frac{1}{\alpha_{\mathcal{UV}}} \mathcal{U}_\omega^{out}(x)\mathcal{V}_\omega^{in\,*}(x') \right\} \\
&+ \ \theta(\tau'-\tau) \int_{-\infty}^0 d\omega \frac{1}{\alpha_{\mathcal{UV}}} \mathcal{U}_\omega^{in}(x)\mathcal{V}_\omega^{out\,*}(x')
\end{aligned}
\qquad (5.14)
$$

and the part which encodes the pairs production is given by :

$$\mathcal{R}e \ \mathrm{tr} \ \mathcal{G}(x,x) \ = \ \mathcal{I}m \int d^2x \int_{-\infty}^0 d\omega \frac{1}{\alpha_{\mathcal{UV}}} \mathcal{U}_\omega^{out}(x)\mathcal{V}_\omega^{in\,*}(x) \qquad . \qquad (5.15)$$

Using the integral representation (given in Appendix **C**) of the product of modes appearing in this last equation we get, once the space-time volume element is reexpressed as $d^2x = a\rho d\rho d\tau$:

$$
\begin{aligned}
\mathcal{R}e \ \mathrm{tr} \ \mathcal{G}(x,x) \ &= \ \mathcal{I}m \int d\tau \int_{-\infty}^0 d\omega \frac{1}{4\pi} e^{-\frac{\pi m^2}{2qE}} e^{\frac{\pi\omega}{2a}} \int_0^\infty \rho d\rho e^{-\epsilon\rho^2} \\
&\qquad \int_{-\infty}^\infty \frac{du}{\cosh u} e^{i(\frac{\omega}{a}-\frac{m^2}{qE})u} e^{i\frac{qE}{2}\rho^2 \tanh u} I_{i\frac{\omega}{a}}\left(\frac{qE\rho^2}{2\cosh u}\right)
\end{aligned}
\qquad (5.16)
$$

This representation allows us to perform the integration on m^2 and we get for the imaginary part of the functional $\mathcal{W}$:

$$\mathcal{I}m\mathcal{W} = -\frac{qE}{4\pi}\mathcal{I}m \int d\tau \int_{-\infty}^0 d\omega \int_{-\infty-i\frac{\pi}{2}}^{\infty-i\frac{\pi}{2}} \frac{d\theta}{\theta \sinh\theta} e^{-i\frac{m^2}{qE}\theta} e^{i\frac{\omega}{a}\theta} \int_0^\infty \rho d\rho e^{-\epsilon\rho^2} e^{i\frac{qE}{2}\rho^2 \coth\theta} e^{\frac{\pi\omega}{2a}} J_{i\frac{\omega}{a}}\left(\frac{qE\rho^2}{2\sinh\theta}\right) \qquad . \qquad (5.17)$$

where we have introduced the new variable $\theta = u - i\frac{\pi}{2}$ and rewrite an Bessel-I function as a J function. The next step consists now to carry out the ρ^2 integration which, thanks to the formula (6.611.1) of [13], leads us to :

$$\mathcal{I}m\mathcal{W} \ = \ \mathcal{I}m \frac{e^{-i\frac{\pi}{2}}}{4\pi} \int d\tau \int_{-\infty}^0 d\omega \int_{-\infty-i\frac{\pi}{2}}^{\infty-i\frac{\pi}{2}} d\theta \frac{\mathrm{sgn}(\mathcal{R}e\theta)}{\theta \sinh\theta} e^{-i\frac{m^2}{qE}\theta} e^{i\frac{\omega}{a}\theta} \{\mathrm{sgn}(\mathcal{R}e\,\theta) \sinh\theta + \cosh\theta\}^{i\frac{\omega}{a}} \qquad .$$

20

Finally, by splitting the θ integral according to the sign of $\mathcal{R}e\,\theta$ we obtain:

$$
\begin{aligned}
\mathcal{I}m\mathcal{W} &= \frac{1}{8\pi}\int d\tau \int_{-\infty}^{\infty}\frac{d\theta}{\theta}\int_{-\infty}^{0}d\omega \frac{e^{-i\frac{m^2}{qE}\theta}e^{2i\frac{\omega}{a}\theta}}{\sinh\theta} - \frac{1}{8\pi}\int d\tau \int_{-\infty}^{\infty}\frac{d\theta}{\theta}\int_{-\infty}^{0}d\omega\frac{e^{-i\frac{m^2}{qE}\theta}}{\sinh\theta} \\
&= \frac{T}{4\pi}\int_{-\infty}^{0}d\omega\,\ln(1+e^{-\frac{\pi m^2}{qE}}) - \frac{T}{4\pi}\int_{-\infty}^{0}d\omega\,\ln(1+e^{-\frac{\pi m^2}{qE}}e^{2\pi\frac{\omega}{a}})
\end{aligned}
\tag{5.18}
$$

in accord with our result (4.30) established by using the Bogoljubov transformation coefficient between *in*- and *out*-modes.

5.2   Schwinger representation of the Feynman propagator

For the sake of completeness, let us briefly recall the so-called Schwinger representation of the Feynman propagator. Formally, the Feynman propagator G_F is the inverse of the kinetic operator $K = \frac{\delta^d(x-y)}{\sqrt{-g(x)}}\{-\mathcal{D}_\mu\mathcal{D}^\mu + (m^2 - i\epsilon)\}$ and can thus be computed as $G_F = -K^{-1} = -i\int_0^\infty ds\,e^{-iKs}$. To express it, Schwinger [4] introduces the kernel defined as $K(x,y;s) = \langle x'|e^{-iKs}|x\rangle$ which obeys a Schrödinger equation :

$$
(\mathcal{D}_\mu\mathcal{D}^\mu - m^2 + i\frac{\partial}{\partial s})K(x,y;s) = 0
\tag{5.19}
$$

with the boundary condition $K(x,y;s\to 0^+) = \delta^d(x-y)$. Using this proper time representation of the Feynman propagator

$$
G_F(x,x') = \int_0^\infty ds\,K(x,x';s)\quad ,
\tag{5.20}
$$

the vacuum persistence amplitude can be written as:

$$
\mathcal{I}m\mathcal{W} = \mathcal{I}m\int_{m^2}^\infty dm^2 \int d^dx \int_0^\infty ds\,K(x,x;s)\quad .
\tag{5.21}
$$

The kernel $K(x,x';s)$, defined as the matrix elements of the evolution operator K, can be expressed as a path integral:

$$
K(x,x';s) = \int \mathcal{D}X^\mu(s')e^{iS(x,x';X^\mu(s');s)}
\tag{5.22}
$$

where the domain of integration covers all the paths that connect the point x to the point x' in a (rescaled) time[8] s, and $S(x,x';x^\mu(s');s)$ is the classical action along these trajectories, given by:

$$
S(x,x';X^\mu(s');s) = \int_0^s ds\left\{\frac{\dot{X}^2}{4} + q\dot{X}^\mu A_\mu(X[s]) - m^2\right\}\quad .
\tag{5.23}
$$

Moreover, let us emphasized that if it is often supposed that the Feynman propagator obtained from modes coincides with the Schwinger representation (5.20) displayed here above, it is only in a few cases that their equivalence has been proved (See for example [14, 15]).

---

[8]This time variable is proportional to the proper time measured along the natural trajectory that connects x to x', but, in general, does not coincide with it.

Reduction of functional integrals

The Schwinger propagator introduced above via a functional integral is a formal object that has to be defined more accurately. To this end we remind the Feynman prescription [16]. We divide the time interval s into N subintervals of length ϵ, such that $s = N\epsilon$, and introduce N times the closure relation in the expression of $K(x, x'; s)$ taken as the matrix element:

$$
\begin{aligned}
K(x, x'; s) &= \langle t', z' | e^{-i\hat{H}s} | t, z \rangle \\
&= \int \left(\prod_{i=1}^{N} dz_i dt_i \right) \langle x' | e^{-iH\epsilon} | x_N \rangle \ldots \langle x_1 | e^{-i\hat{H}\epsilon} | x \rangle \\
&= \int \left(\prod_{i=1}^{N} dz_i dt_i \right) \prod_{j=0}^{N} \left\{ \delta(t_{j+1} - t_j) \delta(z_{j+1} - z_j) - i\epsilon \langle x_{j+1} | \hat{H} | x_j \rangle \right\}, \quad (5.24)
\end{aligned}
$$

before taking the limit $N \mapsto \infty$ at the end of the calculation.

Let us first consider the problem for an inertial observer. Here the hamiltonian operator reads:

$$
\hat{H} = \left\{ (\partial_t - iqA_t)^2 - \partial_z^2 + m^2 \right\} = \left\{ -(\hat{p}_t - qA_t)^2 + \hat{p}_z^2 + m^2 \right\}. \quad (5.25)
$$

We compute its matrix elements in position representation, with $|x\rangle = |z\rangle \otimes |t\rangle$. Using the orthonormalized eigenvectors of the momentum operators $\hat{p}_z |k\rangle = k |k\rangle$ and $\hat{p}_t |\omega\rangle = -\omega |\omega\rangle$ which connect to the position eigenvector by $\langle k | z \rangle = \frac{e^{-ikz}}{\sqrt{2\pi}}$ and $\langle t | \omega \rangle = \frac{e^{-i\omega t}}{\sqrt{2\pi}}$. Thus,

$$
\begin{aligned}
\langle x_{j+1} | \hat{H} | x_j \rangle &= -\delta(z_{j+1} - z_j) \int d\omega \langle t_{j+1} | \omega \rangle \langle \omega | (\hat{p}_t - qA_t(z_j))^2 | t_j \rangle \\
&+ \delta(t_{j+1} - t_j) \int dk \langle z_{j+1} | k \rangle \langle k | \hat{p}_z^2 | z_j \rangle + \delta(z_{j+1} - z_j) \delta(t_{j+1} - t_j) m^2 \\
&= -\delta(z_{j+1} - z_j) \int \frac{d\omega}{2\pi} (\omega + qA_t(z_j))^2 e^{-i\omega(t_{j+1} - t_j)} \\
&+ \delta(t_{j+1} - t_j) \int \frac{dk}{2\pi} k^2 e^{ik(z_{j+1} - z_j)} + \delta(z_{j+1} - z_j) \delta(t_{j+1} - t_j) m^2 \\
&= \int dk \, d\omega \frac{e^{ik(z_{j+1} - z_j)}}{2\pi} \frac{e^{-i\omega(t_{j+1} - t_j)}}{2\pi} H_{cl}(k, \omega, z_j) \quad (5.26)
\end{aligned}
$$

where we have introduced the classical Routh function $H_{cl}(k, \omega, z_j) = -(\omega + qA_t(z_j))^2 + k^2 + m^2$. Inserting this in eq. (5.24), the kernel $K(x, x'; s)$ reads as:

$$
\begin{aligned}
K(x, x'; s) &= \int (\prod_{j=1}^{N} dz_j dt_j) \prod_{j=0}^{N} \int dk_j \, d\omega_j \frac{e^{ik_j(z_{j+1} - z_j)}}{2\pi} \frac{e^{-i\omega_j(t_{j+1} - t_j)}}{2\pi} e^{-i\epsilon H_{cl}(k_j, \omega_j, z_j)} \\
&= \int (\prod_{j=0}^{N} \frac{dk_j}{2\pi}) \int \frac{d\omega_0}{2\pi} e^{-i\omega_0(t' - t)} \int \prod_{j=1}^{N} dz_j e^{ik_j \dot{z}_j \epsilon} e^{-i\epsilon H_{cl}(k_j, \omega_0, z_j)} \\
&= \int \prod_{i=1}^{N} dz_i \int \frac{d\omega_0}{2\pi} e^{-i\omega_0(t' - t)} (4\pi i\epsilon)^{-\frac{N+1}{2}} e^{i\epsilon \dot{z}_j^2 / 4} e^{i\epsilon(\omega + qEz_j)^2} e^{-i\epsilon m^2} \\
&= \int \frac{d\omega}{2\pi} e^{-i\omega(t' - t)} K_\omega(z, z'; s) \quad (5.27)
\end{aligned}
$$

22

where the last equality defines the Fourier transform of the kernel $K(x, x'; s)$ as the path integral

$$K_\omega(z, z'; s) = \int \mathcal{D}z(s) e^{i \int_0^s \mathcal{L}_\omega(z, \dot{z}) ds'}, \tag{5.28}$$

with respect to the (classical) Lagrangian $\mathcal{L}_\omega(z, \dot{z}) = \dot{z}^2/4 + (\omega + qEz)^2 - m^2$.

A similar calculation can be done in Rindler coordinates $(\tau, \rho = a^{-1} \exp(a\xi))$, where we use the τ independence of the hamiltonian to perform the reduction of the path integral to a one dimension quantum mechanical problem. However, due to the explicit coordinate dependence of the metric, the calculation do not reduce just to a strict rewriting of the previous one. The Schwinger kernel is now given by :

$$
\begin{aligned}
K(x, x'; s) &= \langle \rho', \tau' | e^{-i\hat{H}s} | \rho, \tau \rangle \\
&= \frac{1}{\sqrt{a^2 \rho \rho'}} \int \left(\prod_{i=1}^N d\rho_i d\tau_i \right) \prod_{j=0}^N \left\{ \delta(\rho_{j+1} - \rho_j) \delta(\tau_{j+1} - \tau_j) - i\epsilon \langle x_{j+1} | (a\hat{\rho})^{\frac{1}{2}} \hat{H} (a\hat{\rho})^{\frac{1}{2}} | x_j \rangle \right\}
\end{aligned}
$$

The appearance of extra $\rho^{\frac{1}{2}}$ factors results from the coordinate dependence of the volume element. Here the hamiltonian reads as :

$$\hat{H} = \hat{p}_\rho^2 - (\hat{p}_\tau - \frac{aqE}{2}\rho^2)^2 \frac{1}{a^2 \rho^2} - \frac{1}{4\rho^2} + m^2 \tag{5.29}$$

where the momentum operators are $\hat{p}_\tau = \frac{1}{i}\partial_\tau$ and $\hat{p}_\rho = \frac{1}{i}(\partial_\rho + \frac{1}{2\rho})$. Compared to the classical hamiltonian, an extra quantum term $1/4\rho^2$ appears, just as for the plane rotator [15]. This is dictated both by the requirement of hermiticity with respect to the measure $\rho \, d\rho$ and the classical commutation relations (see refs [17]). The orthonormalized eigenstates of these operators and the connectors between them and the position eigenvectors $(|x\rangle = |\rho\rangle \otimes |\tau\rangle$, with $\langle \rho' | \rho \rangle = (a\rho)^{-1}\delta(\rho' - \rho))$ are:

$$\hat{p}_\rho | k \rangle = k | k \rangle \quad , \quad \hat{p}_\tau | \omega \rangle = -\omega | \omega \rangle \quad ,$$

$$\langle k | \rho \rangle = (a\rho)^{-\frac{1}{2}} \frac{e^{-ik\rho}}{\sqrt{2\pi}} \quad , \quad \langle \tau | \omega \rangle = \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} \quad . \tag{5.30}$$

So we obtain the matrix elements:

$$
\begin{aligned}
\langle x_{j+1} | (a\rho)^{\frac{1}{2}} \hat{H} (a\rho)^{\frac{1}{2}} | x_j \rangle &= (a^2 \rho_j \rho_{j+1})^{\frac{1}{2}} \langle x_{j+1} | \hat{H} | x_j \rangle \\
&= -\frac{1}{a^2 \rho_j^2} \delta(\rho_{j+1} - \rho_j) \langle \tau_{j+1} | (\hat{p}_\tau - qEA_\tau(\xi))^2 | \tau_j \rangle \\
&\quad + \delta(\tau_{j+1} - \tau_j) \int dk \frac{e^{ik(\rho_{j+1} - \rho_j)}}{2\pi} k^2 + (m^2 - \frac{1}{4\rho_j^2}) \delta(\tau_{j+1} - \tau_j) \delta(\rho_{j+1} - \rho_j) \\
&= \int dk d\omega \frac{e^{ik(\rho_{j+1} - \rho_j)}}{2\pi} \frac{e^{-i\omega(\tau_{j+1} - \tau_j)}}{2\pi} H_{cl}(k, \omega, \rho_j) \tag{5.31}
\end{aligned}
$$

where we have set $H_{cl}(k, \omega, \rho) = \left\{ -\frac{1}{a^2 \rho^2}(\omega + \frac{aqE\rho^2}{2})^2 + k^2 - \frac{1}{4\rho^2} \right\} + m^2$. Inserting, the matrix elements (5.31) into the expression of the kernel (5.29) we obtain :

$$
\begin{aligned}
K(x, x'; s) &= (a^2 \rho \rho')^{-\frac{1}{2}} \int \prod_{i=1}^N d\rho_i d\tau_i \int \prod_{j=0}^N dk_j d\omega_j \frac{e^{ik_j(\rho_{j+1} - \rho_j)}}{2\pi} \frac{e^{-i\omega_j(\tau_{j+1} - \tau_j)}}{2\pi} e^{-i\epsilon H_{cl}(k_j, \omega_j, \rho_j)} \\
&= (a^2 \rho \rho')^{-\frac{1}{2}} \int \prod_{i=1}^N d\rho_i \int \frac{d\omega_0}{2\pi} e^{-i\omega_0(\tau' - \tau)} \prod_{j=0}^N \frac{dk_j}{2\pi} e^{ik_j \dot{\rho}_j \epsilon} e^{-i\epsilon H_{cl}(k_j, \omega_0, \rho_j)} \quad . \tag{5.32}
\end{aligned}
$$

As in the calculation done in minkowskian coordinates, the independence of the hamiltonian with respect to the time variable makes its conjugate momentum conserved. Each τ_j integration generates a delta function of $\omega_j - \omega_{j-1}$, making all ω_j integration trivial but the first one on ω_0. Moreover the integrals on the k_j are gaussian, and we easily obtain:

$$
\begin{aligned}
K(x, x'; s) &= \frac{1}{(a^2 \rho \rho')^{\frac{1}{2}}} (4\pi i \epsilon)^{-\frac{N+1}{2}} \int \frac{d\omega_0}{2\pi} e^{-i\omega_0(\tau'-\tau)} \int \prod_{i=1}^{N} d\rho_i \prod_{j=0}^{N} e^{i\epsilon \dot{\rho}_j^2/4} e^{i\epsilon \frac{1}{\rho_j^2}(\omega_0 + qEA_\tau(\rho_j))^2} e^{-i(m^2 - \frac{1}{4\rho_j^2})\epsilon} \\
&\equiv \frac{1}{(a^2 \rho \rho')^{\frac{1}{2}}} \int \frac{d\omega}{2\pi} e^{-i\omega(\tau'-\tau)} K_\omega(\rho, \rho'; s)
\end{aligned}
\tag{5.33}
$$

where, as for the inertial propagator, we have introduced a Fourier transform with respect to the Rindler τ time as the path integral :

$$
K_\omega^{Rin}(\rho, \rho'; s) = \int \mathcal{D}\rho \exp i \int_0^s \mathcal{L}_\omega(\rho, \dot{\rho}) ds' \quad ; \tag{5.34}
$$

the lagrangian being now given by :

$$
\mathcal{L}_\omega(\rho, \dot{\rho}) = \dot{\rho}^2/4 + \frac{1}{a^2 \rho^2}(\omega + q\frac{E}{2}a\rho^2)^2 + \frac{1}{4\rho^2} - m^2 \quad . \tag{5.35}
$$

To pursue our analysis we have to evaluate the two (unidimensional) functional integrals (5.28) and (5.34). The first one can be evaluated explicitly, but we did not succeed in expressing the second in a close unambiguous form. To go ahead, we shall evaluate them at one-loop approximation, which actually is exact for gaussian integrals and thus gives the correct answer for the functional integral referring to minkowskian coordinates (5.28). In this approximation we just have (in principle) to compute the classical action S_{cl}, as the well known Pauli–Van Vleck formula [18, 19] gives :

$$
K(\mathbf{q}, t; \mathbf{q}', t') = \int \mathcal{D}\mathbf{q}(t) \exp i S(\mathbf{q}(t)) = \left(\frac{1}{2\pi i} \det \frac{\partial^2 S_{cl}}{\partial \mathbf{q}' \partial \mathbf{q}}\right)^{\frac{1}{2}} \exp i S_{cl} \quad . \tag{5.36}
$$

In principle this formula necessitates the evaluation of the hessian matrix $\partial^2_{qq'} S_{cl}$. At first sight this implies the knowledge of the general two point expression of the action. But actually it suffices to know the general classical solution of the equation of motion to obtain it. Indeed it can be shown (see for instance [25]) that the hessian matrix is the inverse of the matrix built on Jacobi fields vanishing at point $\mathbf{q}$. These are simply obtained by varying the general solution of the equation of motion with respect to the initial velocity components[9][19, 27].
Let us apply this method to compute the kernels (5.28 and 5.34) relevant to the evaluation of trace of the Green function and rate of vacuum decays. Here we have only to take into account closed paths in the functional integrals, starting and ending from the same value of the z or the ρ coordinate.
In z-coordinate the calculation is straightforward. The classical motion equation is:

$$
\frac{d^2 z}{ds'^2} - 4q^2 E^2 (z + \frac{\omega}{qE}) = 0 \tag{5.37}
$$

---

[9]The reader will easily recognize in this method the n-dimensional generalization of the method advocated by S. Coleman to compute functional determinant [26].

whose particular solution such that $z(s' = \mp \frac{s}{2}) = z$ is given by :

$$\bar{z}(s') = -\frac{\omega}{qE} + (z + \frac{\omega}{qE})\frac{\cosh 2qEs'}{\cosh qEs} \qquad . \tag{5.38}$$

The value of the classical action computed along this closed trajectory reads:

$$\begin{aligned} S_{cl} &= \int_{-\frac{s}{2}}^{\frac{s}{2}} ds' \, \mathcal{L}_\omega(z, \dot{z})|_{z=\bar{z}(s')} \\ &= qE(z + \frac{\omega}{qE})^2 \frac{\cosh 2qEs - 1}{\sinh 2qEs} - m^2 s \qquad . \end{aligned} \tag{5.39}$$

Here the computation of Jacobi fields is immediate, because for quadratic potential it only consists in solving a harmonic oscillator problem :

$$\left\{ \frac{d^2}{d^2 s'} - 4q^2 E^2 \right\} h(s') = 0 \qquad . \tag{5.40}$$

whose solution obeying the Cauchy conditions $h(s' = -\frac{s}{2}) = 0$ and $\frac{\partial}{\partial s'}h(s')|_{s'=-\frac{s}{2}} = 2$ (because the unusual normalization $1/4$ of the kinetic part of the lagrangian) is:

$$h(s') = \frac{1}{qE} \sinh 2qE(s' + \frac{s}{2}) \qquad . \tag{5.41}$$

Thus we get:

$$K_\omega(z, z; s) = (\frac{qE}{2\pi i \sinh 2qEs})^{\frac{1}{2}} e^{iqE(z+\frac{\omega}{qE})^2 \frac{\cosh 2qEs-1}{\sinh 2qEs} - im^2 s} \qquad . \tag{5.42}$$

Inserting this expression in eq. (5.21), the vacuum amplitude persistence, after t and m^2 integration ($T = \int dt$), amount to :

$$\begin{aligned} \mathcal{I}m\,\mathcal{W} &= \mathcal{I}m\frac{1}{i}(\frac{qE}{2i\pi})^{\frac{1}{2}}\frac{T}{2\pi} \int d\omega \int_0^\infty \frac{ds}{s} e^{-im^2 s} \int \frac{dz}{(\sinh 2qEs)^{\frac{1}{2}}} e^{iqE\frac{\sinh qEs}{\cosh qEs}(z+\frac{\omega}{qE})^2} \\ &= \frac{T}{4\pi} \int d\omega \ln(1 + e^{-\pi\frac{m^2}{qE}}) \qquad . \end{aligned} \tag{5.43}$$

This one loop computation gives the exact result, as expected, because here the lagrangian is quadratic and the one-loop approximation (5.42) is actually exact.

Let us turn now to the same problem formulated in Rindler coordinates, for the accelerated observer. We have to evaluate the functional integral (5.34) on closed paths starting from and ending on $\rho(s' = -\frac{s}{2}) = \rho(s' = \frac{s}{2}) = \rho$. The equation of motion, derived from the lagrangian (5.35) is:

$$\frac{1}{2}\frac{d^2\rho}{ds'^2} + \frac{2}{\rho^2}(\frac{\omega^2}{a^2} + \frac{1}{4}) - \frac{1}{2}q^2 E^2 \rho = 0 \tag{5.44}$$

which, by use of the energy theorem, reduces to

$$\frac{1}{q^2 E^2} \left(\frac{d\rho}{ds'}\right)^2 = 2\beta + (\rho^2 + \frac{\tilde{\Omega}^2}{\rho^2}) \tag{5.45}$$

where β is an integration constant and

$$\tilde{\Omega} = \frac{2}{qE}\sqrt{\frac{\omega^2}{a^2} + \frac{1}{4}} > 0 \qquad , \tag{5.46}$$

the correspondence between this quantum number and the classical constant of motion being $\tilde{\Omega} \approx (m/qE)^2 \Omega$. When $\beta > 0$, obviously there is no turning points in ρ; the motion goes from $\rho = 0$ to $\rho = \infty$, and the trajectory never crosses twice the same point. Actually, to get a trajectory with a bounce on the potential barrier, β must be less than $-\tilde{\Omega}$. Hereafter we restrict ourselves to this class of trajectories. Integrating equation (5.45), we obtain as solution of equation (5.44) :

$$\rho^2(s') + \beta = \varepsilon(\rho_0^2 + \beta)\frac{\cosh\{2qE(s' - s_0) + \varphi\}}{\cosh\varphi} \tag{5.47}$$

in terms of arbitrary constants ρ_0, β and s_0 and where $\varepsilon = \pm 1$ and we have set, for further notational convenience, $\cosh\varphi = (\rho_0^2 + \beta)/\sqrt{\beta^2 - \tilde{\Omega}^2}$. The solution which bounces in time s, such that $\rho(s' = -\frac{s}{2}) = \rho(s' = \frac{s}{2}) = \rho$ is:

$$\bar{\rho}^2(s') + \bar{\beta} = (\rho^2 + \bar{\beta})\frac{\cosh 2qEs'}{\cosh\bar{\varphi}} \tag{5.48}$$

where the value of β is now fixed to $\bar{\beta} = \left\{\rho^2 - \cosh(qEs)\sqrt{\rho^4 + \tilde{\Omega}^2\sinh^2(qEs)}\right\}/\sinh^2(qEs)$, which implies that $\bar{\varphi} = -(qEs)$. Obviously $\bar{\beta}$ is negative and $\rho^2 + \bar{\beta}$ will be negative or positive according to ρ^2 is less or greater than $\tilde{\Omega}$ i.e. according to the position of ρ with respect to $(\sqrt{\tilde{\Omega}})$, where the minimum of the potential $\rho^2 + \tilde{\Omega}^2/\rho^2$ occurs. The computation of the value of the classical action along this trajectory is straightforward. We obtain, in terms of the variable ψ, defined through the relation $\sinh\psi = \tilde{\Omega}\sinh qEs/\rho^2$:

$$S'_\omega(\rho, s) = \frac{\sqrt{\omega^2 + a^2}}{a}\left(\frac{\cosh qEs - \cosh\psi}{\sinh\psi} + \psi\right) \qquad , \tag{5.49}$$

on which one it is easy to check that:

$$\frac{\partial S'_\omega(\rho, s)}{\partial s} = -\frac{q^2 E^2 \beta}{2} \qquad . \tag{5.50}$$

It remains now to compute the Jacobi field $h(s')$ obeying the Cauchy conditions $h(s' = -\frac{s}{2}) = 0$ and $\partial_{s'} h(s')|_{s'=-\frac{s}{2}} = 2$. Varying the general solution (5.47) of the equation of motion with respect to the "energy" parameter β we get a Jacobi field vanishing for $s' = -\frac{s}{2}$. Once normalized such that $\partial'_s \tilde{h}(s')\big|_{s'=-\frac{s}{2}} = 2$, its value for $s' = s/2$ becomes:

$$h(\frac{s}{2}) = \frac{2}{qE}\sinh qEs \cosh\psi \tag{5.51}$$

and gives the value of the Van Vleck determinant. Collecting all these results, we obtain for the path integral (5.34) with $\rho = \rho'$, the approximated expression :

$$K^{Rin}_\omega(\rho, \rho; s) \approx \frac{e^{-im^2 s + i\frac{qE\omega}{a}s + i\frac{\sqrt{\omega^2 + a^2}}{a}\left(\frac{\cosh qEs - \cosh\psi}{\sinh\psi} + \psi\right)}}{\sqrt{\frac{4i\pi}{qE}\sinh qEs \cosh\psi}} \qquad , \tag{5.52}$$

and for the vacuum persistence amplitude :

$$\mathcal{I}m\,\mathcal{W} \;=\; \mathcal{I}m\frac{T}{2\pi i}\int d\omega \int_0^\infty \frac{ds}{s}\, e^{-im^2 s+i\frac{qE\omega}{a}s}\sqrt{\frac{qE\tilde{\Omega}}{16 i\pi}}\int_0^\infty d\psi \frac{(\cosh\psi)^{\frac{1}{2}}}{(\sinh\psi)^{\frac{3}{2}}}e^{i\frac{\sqrt{\omega^2+a^2}}{a}\left(\frac{\cosh\theta-\cosh\psi}{\sinh\psi}+\psi\right)} \qquad .$$

$$(5.53)$$

Let us evaluate the ψ integral by the saddle point method. Indeed, given the approximations used to evaluate $K_\omega^{Rin}(\rho,\rho;s)$, it would be inconsistent to endeavor to go beyond this approximation. The phase reaches its maximum (with respect to ψ) at $\psi = qEs$, corresponding to a saddle point of width $(\sqrt{\omega^2+a^2}/a)\coth qEs$. This leads to the gaussian approximation :

$$\int_0^\infty d\psi \frac{(\cosh\psi)^{\frac{1}{2}}}{(\sinh\psi)^{\frac{3}{2}}}e^{i\frac{\sqrt{\omega^2+a^2}}{a}\left(\frac{\cosh\theta-\cosh\psi}{\sinh\psi}+\psi\right)} \approx \sqrt{\frac{2i\pi a}{\omega^2+a^2}}\frac{e^{i\frac{\sqrt{\omega^2+a^2}}{a}qEs}}{\sinh qEs} \qquad (5.54)$$

and the vacuum persistence amplitude reads:

$$\mathcal{I}m\,\mathcal{W} \;\approx\; \mathcal{I}m\frac{T}{4\pi i}\int d\omega \int_0^\infty \frac{ds}{s\sinh qEs}\, e^{-im^2 s}e^{i\frac{\omega}{a}qEs}e^{i\frac{\sqrt{\omega^2+a^2}}{a}qEs} \qquad (5.55)$$

$$\approx\; \mathcal{I}m\frac{T}{4\pi i}\int d\omega \int_0^\infty \frac{ds}{s\sinh qEs}\left(\theta(\omega)e^{-im^2 s+i2\frac{\omega}{a}qEs}+\theta(-\omega)e^{-im^2 s}\right) \qquad . \;(5.56)$$

Comparing eq. (5.56) with eq. (5.18) we see that their difference reduces to

$$\mathcal{R}e\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty}\frac{e^{i(2\omega\,\theta-\mu^2\theta)}}{\theta\,\sinh\theta}\,d\theta = -\frac{1}{2\,\epsilon^2} \qquad (5.57)$$

once we take into account the $i\,\epsilon$ prescription. This divergent quantity is physically meaningless. It is independent of the physical parameters of the problem and must be subtracted in order to recover the stability of the Rindler vacuum in the absence of the electric field. Of course all the finite corrections we have mentioned before, and that we recover here above, are mathematically meaningless; the approximations we used for the Schwinger kernel making them all irrelevant. Our motivation for having discussed them is just to show that no spurious term are introduced in the dominant contribution to $\mathcal{I}m\,\mathcal{W}$. Actually this leading order can be obtained directly by ignoring all subtleties but just using a quadratic approximation of the potential :

$$V(\rho) \;=\; \frac{1}{\rho^2}(\frac{\omega}{a}+\frac{qE}{2}\rho^2)^2 + \frac{1}{4\rho^2} - m^2$$

$$\approx\; (\frac{2\omega qE}{a}-m^2) + q^2 E^2(\rho-\rho_0)^2 \qquad (5.58)$$

around the minimum $\rho_0^2 = \tilde{\Omega} \approx \frac{2}{aqE}\omega$. The computation is similar to the one discussed for the inertial observer once we have substituted $m^2 - \frac{\tilde{\omega}qE}{a} - \frac{\omega qE}{a}$ to m^2. As for the evaluation of (5.42) we obtain now :

$$K_\omega^{Rin}(\rho,\rho;s) \approx \sqrt{\frac{qE}{2i\pi\sinh 2qEs}}e^{i(\frac{\sqrt{\omega^2+a^2}+\omega qE}{a}-m^2)s}e^{iqE(\rho-\rho_0)^2\frac{\cosh 2qEs-1}{\sinh 2qEs}} \qquad (5.59)$$

whose trace gives expression (5.55), analog to (5.43).

5.2.1 Is the zero winding propagator the Rindler propagator?

In the appendix of ref. [5], the Fourier transform of the inertial Feynman propagator was expressed as a sum of winding terms, similar to those given in ref. [28]. Each term of this sum corresponds the Fourier transform of a path integration restricted to the subclass of paths joining their ends after having performed a fixed number of winding around the common vertex of the four Rindler quadrants (around the origin O, see fig. (**1**)). It was conjectured that the kernel $\tilde{K}_\omega^{w=0}$ obtained from the zero winding sector, provides the kernel leading to the Rindler propagator : $\tilde{K}_\omega^{w=0} = K_\omega^{Rin}$. Hereafter we reinforce this conjecture by showing that the rate calculated from this zero winding kernel :

$$\tilde{K}_\omega^{w=0} = -\frac{1}{4\pi} \frac{e^{-i(m^2 - \frac{\omega}{a})Es}}{\sinh qEs} e^{i\frac{qE}{4}(\rho^2 + \rho'^{\,2})\coth qEs} I_{-i\frac{|\omega|}{a}} \left(-i\frac{qE\rho\rho'}{2\sinh qEs} \right) \tag{5.60}$$

leads to our previous results. To this end we substitute in eq. (5.21) the expression of $K^{Rin}(x, x, s)$ obtained by using the expression (5.60) in eq. (5.33). Reexpressing the Bessel-I function as a sum of Bessel-K functions and integrating over the ρ variable, we obtain :

$$\begin{aligned}
&\mathcal{R}e\frac{\mathcal{I}}{2\pi} \int_0^\infty \rho\, d\rho \int_{-\infty}^\infty d\omega \int_0^\infty \tilde{K}_\omega^{w=0}(\rho, \rho; s)\frac{ds}{s} \\
&= \mathcal{R}e\frac{\mathcal{I}}{2\pi} \int_{-\infty}^\infty d\omega \int_0^\infty e^{-i(m^2 s + \frac{\omega}{a}qe\,s)}\frac{ds}{s} \int_0^\infty \rho\, d\rho \\
&\quad \frac{qE\rho\, e^{i\frac{qE}{2}\rho^2 \coth qEs}}{2\pi\sinh(qE\,s)} \left\{ K_{i\frac{\omega}{a}}\left(i\frac{qe\,\rho^2}{2\sinh(qE\,s)} \right) + \left(2\,\theta(\omega)\sinh(\pi\frac{\omega}{a}) - e^{\pi\frac{\omega}{a}} \right) K_{i\frac{\omega}{a}}\left(-i\frac{qe\,\rho^2}{2\sinh(qE\,s)} \right) \right\}
\end{aligned} \tag{5.61}$$

Formula (**6.611.3**) of ref. [13] provides us the result of the ρ integration. So we obtain for the vacuum persistence amplitude, computed using (5.60):

$$\mathcal{I}m\mathcal{W} = \mathcal{R}e\frac{\mathcal{I}}{4\pi} \int_{-\infty}^\infty \frac{d\omega}{\sinh(\pi\frac{\omega}{a})} \int_0^\infty \frac{ds}{s\sinh(qE\,s)} e^{-im^2 s} \left\{ \theta(\omega)\sinh(\pi\frac{\omega}{a}) e^{i2\frac{\omega}{a}qE\,s} - \theta(-\omega)\sinh(\pi\frac{\omega}{a}) \right\} \tag{5.62}$$

i.e. expression (5.56) which is equivalent to eq. (5.18).

Let us note that the Fourier transform of the Schwinger kernel has to satisfy the equation

$$\left[i\partial_s - m^2 + \partial_\rho^2 + \frac{(qE)^2}{4}\rho^2 + \frac{1}{\rho^2}\left(\frac{\omega^2}{a^2} + \frac{1}{4} \right) \right] \tilde{K}_\omega(\rho, \rho'; s) = 0 \tag{5.63}$$

and the initial condition on the **R** quadrant

$$\lim_{s \to 0} \tilde{K}_\omega(\rho, \rho'; s) = \delta(\rho - \rho') \qquad . \tag{5.64}$$

The expression (5.60) satisfies this equation and the initial condition in the following sense:

$$\lim_{s \to 0} \tilde{K}_\omega^{w=0}(\rho, \rho'; s) = \delta(\rho - \rho') + [2\,\theta(\omega)\sinh(\pi\frac{\omega}{a}) - e^{\pi\frac{\omega}{a}}]\delta(\rho + \rho') \qquad . \tag{5.65}$$

This extra term is not a surprise. It is localized on the boundary of the **R** patch and reflects the correlations between the left and right quadrants that appear when the vacuum decay and that pair creation occurs. Actually the existence on the horizon of a singular contribution to the Rindler (uncharged) scalar field has been put into evidence some time ago by Parentani [29] who has shown that such terms are essential to the validity of the theorem stating that the usual Poincaré invariant vacuum state is the state of minimal energy.

6 Construction of Unruh modes

In this section we shall establish connections between Minkowskian and Rindlerian modes. Instead of working with integral transforms, we find more convenient to diagonalize the Bogoljubov transformation. This is achieved by introducing Unruh modes. They correspond to new basis of the Fock spaces, which share the same quantum numbers as the Rindler modes but that do not mix positive and negative frequencies of the Minkowskian modes. The purpose of this section is to build these modes which allows to easily describes algebraically the Minkowskian particle content of the Rindler vacua. Completeness of Minkowskian and Rindler sets of modes insures each element of one set can be expressed as superposition of members of the other one. To obtain the relation between them we shall, instead of computing directly the overlapping integrals, made use of integral representations of the parabolic cylinder functions (see ref.[13], eq. (**9.241.2**) for the mathematics and [8] for their physical meaning) . Accordingly, we may express the *in*- and *out*-modes (eqs 3.9 and followings in the text) as :

$$
\phi_\sigma^{p\,in}(U,V) \;=\; \frac{1}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \frac{e^{-\frac{\pi m^2}{4qE}} e^{-\frac{i\pi}{4}}}{\sqrt{2\pi}(2qE)^{1/4}} e^{-i\sigma V} e^{-i\frac{qE}{4}V^2} e^{i\frac{qE}{4}UV} e^{-i\frac{\sigma^2}{2qE}}
$$
$$
\int_0^\infty dx\, e^{ix\sqrt{2qE}(\frac{V-U}{2} + \frac{\sigma}{qE} - i\frac{x^2}{2})} x^{-i\frac{m^2}{2qE} - \frac{1}{2}} \tag{6.1}
$$

$$
\phi_\sigma^{a\,in}(U,V) \;=\; \frac{1}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \frac{e^{-\frac{i\pi}{4}} e^{-\frac{\pi m^2}{4qE}}}{\sqrt{2\pi}(2qE)^{1/4}} e^{i\sigma U} e^{-i\frac{qE}{4}U^2} e^{i\frac{qE}{4}UV} e^{-i\frac{\sigma^2}{2qE}}
$$
$$
\int_0^\infty dx\, e^{-ix\sqrt{2qE}(\frac{V-U}{2} + \frac{\sigma}{qE} - i\frac{x^2}{2})} x^{-i\frac{m^2}{2qE} - \frac{1}{2}} \tag{6.2}
$$

where we have used the ingoing and outgoing light-like coordinates $U = t - z$ and $V = t + z$. Using the reflexion operation $U \mapsto -V$ and $V \mapsto -U$, we obtain :

$$
\phi_\sigma^{p\,out}(U,V) \;=\; (\phi_\sigma^{p\,in}(-V,-U))^* \quad ,
$$
$$
\phi_\sigma^{a\,out}(U,V) \;=\; (\phi_\sigma^{a\,in}(-V,-U))^* \quad .
$$

On the other hand, from the integral representations (A.5) of the Whittaker's function, we get the two representations of the Rindler modes (see eqs. A.12):

$$
\mathcal{W}_\epsilon^-(U,V) \;=\; \left(\frac{qE}{4\pi a^2}\right)^{1/2} e^{i\frac{qE}{4}UV} \left(\frac{qEV^2}{2}\right)^{-i\frac{\omega}{2a}} \frac{e^{i\epsilon\frac{\pi}{4}} e^{\epsilon\frac{\pi\omega}{4a}}}{\Gamma(-\frac{im^2}{2qE} + \frac{1}{2})} \tag{6.3}
$$
$$
\int_0^\infty dt\, e^{i\frac{qE}{2}UVt} t^{-i\frac{m^2}{qE} - \frac{1}{2}} (1+t)^{-i\frac{\omega}{a} + i\frac{m^2}{2qE} - \frac{1}{2}}
$$
$$
=\; \left(\frac{qE}{4\pi a^2}\right)^{1/2} e^{i\frac{qE}{4}UV} \left(\frac{qEU^2}{2}\right)^{i\frac{\omega}{2a}} \frac{e^{i\epsilon\frac{\pi}{4}} e^{-\epsilon\frac{\pi\omega}{4a}}}{\Gamma(i\frac{\omega}{a} - \frac{im^2}{2qE} + \frac{1}{2})} \tag{6.4}
$$
$$
\int_0^\infty dt\, e^{i\frac{qE}{2}UVt} t^{i\frac{\omega}{a} - i\frac{m^2}{qE} - \frac{1}{2}} (1+t)^{i\frac{m^2}{2qE} - \frac{1}{2}}
$$
$$
=\; \left(\mathcal{W}_\epsilon^+(U,V)\right)^* \quad , \tag{6.5}
$$

where we have reexpressed the Rindler coordinates ξ and τ in terms of light-like Minkowski coordinates, through the relations $\xi = \frac{1}{2a}\ln(-\epsilon a^2 UV)$ and $\tau = \frac{1}{2a}\ln(-\epsilon\frac{V}{U})$.

Let us now consider a wave packet, denoted hereafter as $\Omega_\omega^{p\,in}$, of inertial in-particles $\phi_\sigma^{p\,in}$:

$$\Omega_{p,in}^\omega = \int_{-\infty}^{\infty} A(\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a})\phi_\sigma^{p\,in}\, d(\frac{\sigma}{\sqrt{qE}}) \qquad . \qquad (6.6)$$

Using the integral representation (6.1), we can rewrite this superposition as the convolution :

$$\frac{e^{\frac{\pi m^2}{8qE}} e^{\frac{i\pi}{8}} e^{-i\frac{qE}{4}V(V-U)}}{M\,\Gamma[\frac{1}{2}-i\frac{m^2}{2qE}]} \int_0^\infty e^{ix\sqrt{2qE}\frac{V-U}{2}-i\frac{x^2}{2}} x^{-i\frac{m^2}{2qE}-\frac{1}{2}} F\left[\frac{\omega}{a}; \sqrt{qE}(-V + x\sqrt{\frac{2}{qE}})\right]\, dx \qquad (6.7)$$

where we introduced a two variable function, F, built from the coefficients of the wavepacket (6.6) through the relation:

$$F\left[\frac{\omega}{a}; \beta\right] = \int_{-\infty}^{+\infty} A(\tilde\sigma, \frac{\omega}{a})e^{i\beta\tilde\sigma}e^{-i\frac{\tilde\sigma^2}{2}}\, d\tilde\sigma \qquad . \qquad (6.8)$$

If we choose the arbitrary function F such as :

$$F\left[\frac{\omega}{a}; \beta\right] = \left(\frac{qE}{2a^2}\right)^{\frac{1}{4}} e^{i\beta^2/4}(\beta/\sqrt{2})^{i(\frac{m^2}{2qE}-\frac{\omega}{a})-\frac{1}{2}} \qquad (6.9)$$

(the choice of the coefficient in front of this expression anticipating a subsequent normalization), eq.(6.7) becomes an integral representation of Rindlerian $\mathcal{V}$ in-modes on **P** and **L** , i.e. $\mathcal{W}_\epsilon^-$ functions (see eq. (A.12)) :

$$\Omega_\omega^{p,in} = \left(\frac{8\pi^2 a^2}{qE}\right)^{\frac{1}{4}} \frac{e^{i\frac{\pi}{8}} e^{\frac{\pi m^2}{8qE}}}{M} e^{-i\epsilon\frac{\pi}{4}} e^{-\epsilon\frac{\pi\omega}{4a}}\, \mathcal{W}_\epsilon^- \qquad . \qquad (6.10)$$

While general arguments of completeness insure the existence of the function $A(\sigma/\sqrt{qE}, \omega/a)$, we need its explicit expression to obtain the continuation of $\Omega_\omega^{p,in}$ on the rest of space-time (in the regions $V > 0$). Taking the inverse Fourier transform (with respect to the variable β) of eq. (6.9), we obtain :

$$A(\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a}) = \left(\frac{qE}{2a^2}\right)^{\frac{1}{4}} \frac{e^{i\frac{\sigma^2}{2qE}}}{\sqrt{2\pi}} \int_0^\infty e^{i\frac{p^2}{2}} e^{-i\sqrt{2}\frac{\sigma}{\sqrt{qE}}p} p^{-i\frac{\omega}{a}+i\frac{m^2}{2qE}-\frac{1}{2}}\, dp \qquad (6.11)$$

$$= \left(\frac{qE}{2a^2}\right)^{\frac{1}{4}} \frac{e^{\frac{\pi}{2}(\frac{\omega}{2a}-\frac{m^2}{4qE})} e^{\frac{i\pi}{8}}}{\sqrt{2\pi}} \Gamma[i(\frac{m^2}{2qE}-\frac{\omega}{a})+\frac{1}{2}) D_{i(\frac{\omega}{a}-\frac{m^2}{2qE})-\frac{1}{2}}(\sqrt{2}e^{\frac{3i\pi}{4}}\frac{\sigma}{\sqrt{qE}}) \qquad .$$

Using twice the integral representations (6.11) of the parabolic cylinder functions, the integrals over $\frac{\sigma}{\sqrt{qE}}$ and x become straightforward. We obtain an expression involving only $\mathcal{W}_\epsilon^+$ functions, i.e. modes $\mathcal{U}_{in,R}$ and $\mathcal{U}_{out,F}$:

$$\Omega_\omega^{p,in} = \left(\frac{8\pi^2 a^2}{qE}\right)^{\frac{1}{4}} \frac{e^{i\frac{\pi}{8}} e^{\frac{\pi m^2}{8qE}} \Gamma[i(\frac{m^2}{2qE}-\frac{\omega}{a})+\frac{1}{2}]}{M\,\Gamma[\frac{1}{2}-i\frac{m^2}{2qE}]} e^{i\epsilon\frac{\pi}{4}} e^{\epsilon\frac{\pi\omega}{4a}}\, \mathcal{W}_\epsilon^+ \qquad . \qquad (6.12)$$

30

More explicitly, by introducing the characteristic functions $\chi_{L,R,P,F}$ (which are equal to zero or to one according to that the coordinates considered correspond to a point belonging to the sector labeling the function or not), we may write :

$$
\begin{aligned}
\Omega_\omega^{p,in} &= \chi_P \, \mathcal{V}_{in,P}^\omega - i \, e^{-\frac{\pi m^2}{4qE}} \left(\frac{\cosh[\pi(\frac{\omega}{a} - \frac{m^2}{2qE})]}{|\sinh(\frac{\pi\omega}{a})|} \right)^{\frac{1}{2}} \left[\chi_L \, \mathcal{V}_{in,L}^\omega + i \, \chi_R \, e^{\frac{\pi\omega}{2a}} \frac{\Gamma[i(\frac{m^2}{2qE} - \frac{\omega}{a}) + \frac{1}{2}]}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \mathcal{U}_{in,R}^\omega \right] \\
&\quad - i \, \chi_F \, e^{-\frac{\pi\omega}{2a}} \frac{\Gamma[i(\frac{m^2}{2qE} - \frac{\omega}{a}) + \frac{1}{2}]}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \mathcal{U}_{out,F}^\omega \quad .
\end{aligned} \tag{6.13}
$$

The mode (6.13) so constructed defines an Unruh mode. Indeed it is a (superposition of) positive frequency modes, algebraically related to Rindler modes. Similarly, we may define other orthonormalized *in*-Unruh modes through wave superpositions built as in eq. (6.6) :

$$
\begin{aligned}
\varpi_{p\,in}^\omega &= \int_{-\infty}^\infty A(-\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a}) \phi_\sigma^{p\,in} \, d(\frac{\sigma}{\sqrt{qE}}) = e^{i\psi} \, \mathrm{sgn}(\omega) \left[\chi_R \mathcal{V}_{in,R}^\omega + i\chi_F e^{\frac{\pi\omega}{2a}} \mathcal{V}_{in,F}^\omega \right] \quad , \tag{6.14} \\
\Omega_{a\,in}^\omega &= \int_{-\infty}^\infty A(-\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a}) \phi_\sigma^{a\,in} \, d(\frac{\sigma}{\sqrt{qE}}) \\
&= \chi_P \mathcal{U}_{in,P}^{\omega\,*} - i \, e^{-\frac{\pi m^2}{4qE}} \left(\frac{\cosh[\pi(\frac{\omega}{a} - \frac{m^2}{2qE})]}{|\sinh(\frac{\pi\omega}{a})|} \right)^{\frac{1}{2}} \left[\chi_R \mathcal{U}_{in,R}^{\omega\,*} + i \, e^{\frac{\pi\omega}{2a}} \frac{\Gamma[i(\frac{m^2}{2qE} - \frac{\omega}{a}) + \frac{1}{2}]}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \chi_L \, \mathcal{V}_{in,L}^{\omega\,*} \right] \\
&\quad + i \, e^{-\frac{\pi\omega}{2a}} \frac{\Gamma[i(\frac{m^2}{2qE} - \frac{\omega}{a}) + \frac{1}{2}]}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \mathcal{V}_{out,F}^{\omega\,*} \quad , \tag{6.15} \\
\varpi_{a\,in}^\omega &= \int_{-\infty}^\infty A(\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a}) \phi_\sigma^{a\,in} \, d(\frac{\sigma}{\sqrt{qE}}) = e^{i\psi} \, \mathrm{sgn}(\omega) \left[\chi_L \mathcal{U}_{in,L}^{\omega\,*} + i\chi_F e^{\frac{\pi\omega}{2a}} \mathcal{U}_{in,F}^{\omega\,*} \right] \quad , \tag{6.16}
\end{aligned}
$$

where the phase ψ is given by :

$$
\psi = \arg \left\{ \frac{\Gamma[i\frac{\omega}{a}]}{\Gamma[i(\frac{m^2}{2qE} + \frac{\omega}{a}) + \frac{1}{2}]} \right\} \quad . \tag{6.17}
$$

Let us emphasize some properties of these Unruh modes. By considering on $\mathcal{I}^-$ the supports of the Rindler modes that define them, it is obvious that these Unruh modes are orthogonal to each other and normalized. This also results from the relation :

$$
\int_{-\infty}^\infty d(\frac{\sigma}{\sqrt{qE}}) A(\sigma,\omega) A^*(\varepsilon\,\sigma,\omega') = \theta(\varepsilon)\delta(\omega' - \omega) \qquad \varepsilon = \pm 1 \quad . \tag{6.18}
$$

that can be easily proved using the integral form (6.11) of the coefficients $A(\sigma,\omega)$. On the other hand, once we realize that these modes have to correspond to Minkowskian particle or antiparticle *in*-modes, starting from $\mathcal{I}_R^-$ or $\mathcal{I}_L^-$, they may be built from purely algebraic considerations. Indeed it suffices to fix their behavior on $\mathcal{I}^-$ and continue them across the horizon by requiring continuity (always in term of wave packets). For instance, suppose we want to start with modes that born on $\mathcal{I}_{P,R}^-$, we have to take on $\mathbf{P}$, $\mathcal{V}_{in,P}$ modes (see fig. (**3**)). Then in order to maintain zero Cauchy data on $\mathcal{I}_{R,P}^-$ and $\mathcal{I}_L^-$ we have to paste them respectively to $\mathcal{U}_{in,R}$ and $\mathcal{V}_{in,L}$ modes. Then it remains to fix the linear combination of modes defined on $\mathbf{F}$

that fit our construction on $\mathcal{H}_R^+$ and $\mathcal{H}_L^+$. So, using the relation (A.7) we may fix the relative weights (and phases) between these modes on each quadrant. And an a priori unexpected property appears: the interferences are such that the particle Unruh mode $\Omega^{p\,in}$ vanishes on $\mathcal{I}_{F,L}^+$ and the antiparticle Unruh mode $\Omega^{a\,in}$ on $\mathcal{I}_{F,R}^+$. This finds its mathematical origin in an analyticity property: the fact that the ϖ Unruh modes can be expressed as linear combination of only functions $\mathcal{M}_\epsilon^\pm$ and the modes Ω as pure combination of $\mathcal{W}_\epsilon^\pm$ functions.

Just as we have introduced an *in* Unruh basis (6.66.16) we may consider a *out* Unruh basis built out from superposition of *out*-particle and antiparticle modes with as coefficient the complex conjugate of those used in the construction of the *in* Unruh basis. Actually these modes are obtained by performing a space-time inversion $[(\tau_F, \xi_F) \leftrightarrow (\tau_P, \xi_P),\ (\tau_R, \xi_R) \leftrightarrow (\tau_L, \xi_L)]$ on the previous modes and taking their complex conjugate. They are given by :

$$\Omega_{p,out}^\omega = \int_{-\infty}^\infty A(\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a})^* \phi_\sigma^{p\,out}\, d(\frac{\sigma}{\sqrt{qE}}) \qquad , \tag{6.19}$$

$$\varpi_{p\,out}^\omega = \int_{-\infty}^\infty A(-\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a})^* \phi_\sigma^{p\,out}\, d(\frac{\sigma}{\sqrt{qE}}) \qquad , \tag{6.20}$$

$$\Omega_{a\,out}^\omega = \int_{-\infty}^\infty A(-\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a})^* \phi_\sigma^{a\,out}\, d(\frac{\sigma}{\sqrt{qE}}) \qquad , \tag{6.21}$$

$$\varpi_{a\,out}^\omega = \int_{-\infty}^\infty A(\frac{\sigma}{\sqrt{qE}}, \frac{\omega}{a})^* \phi_\sigma^{a\,out}\, d(\frac{\sigma}{\sqrt{qE}}) \qquad . \tag{6.22}$$

These modes are related to Rindler *out*-modes by a Bogoljubov transformation given in Appendix **B** and schematically represented on fig. (**4**).

To compute the coefficient of the Bogoljubov transformation between the *in* and *out* Unruh modes, we use the Bogoljubov transformation (3.11) to transfer the relation between modes into relation between connectors linking Rindlerian and Minkowskian modes. For instance we obtain for the modes $\Omega_\omega^{p\,out}$:

$$\Omega_\omega^{p\,out} = \int_\infty^\infty A(\frac{\sigma}{\sqrt{qE}}, \omega)^* (\gamma \phi_\sigma^{p\,in} - \delta^* \phi_\sigma^{a\,in\,*}) d(\frac{\sigma}{\sqrt{qE}}) \qquad . \tag{6.23}$$

Fortunately, the γ and δ coefficients are σ independent and factorize out of the integral. The contribution of the antiparticle modes $\phi_\sigma^{a\,in}$ sums directly (see eq. (6.16)) and gives :

$$e^{i\frac{\pi}{2}} e^{-\frac{\pi m^2}{2qE}} \varpi_\omega^{a\,in\,*} \qquad . \tag{6.24}$$

The summation of the particle modes needs a little bit more work. Thanks to the linear relations connecting parabolic cylinder functions, we get:

$$A(\pm\sigma, \omega)^* = \frac{\Gamma[\frac{1}{2} + i\frac{\omega}{a} - i\frac{m^2}{2qE}]}{\sqrt{2\pi}} \left\{ e^{-i\frac{\pi}{2}} e^{-\frac{\pi m^2}{4qE}} e^{\frac{\pi\omega}{2a}} A(\mp\sigma, \omega) + e^{\frac{\pi m^2}{4qE}} e^{-\frac{\pi\omega}{2a}} A(\pm\sigma, \omega) \right\} \tag{6.25}$$

and the particle modes contribution to eq. (6.23) can be written as a sum of two *in* Unruh modes. So we get:

$$\Omega_\omega^{p\,out} = i\sqrt{q_1}\Omega_\omega^{p\,in} + e^{-\frac{\pi m^2}{2qE}} e^{\frac{\pi\omega}{a}} \sqrt{q_1} e^{i\psi'} \varpi_\omega^{p\,in} + ie^{-\frac{\pi m^2}{2qE}} \varpi_\omega^{a\,in\,*} \qquad , \tag{6.26}$$

where q_1 is defined by eq. (4.14) and ψ' is the phase:

$$\psi' = \arg\left\{\frac{\Gamma[i(\frac{\omega}{a} - \frac{m^2}{2qE}) + \frac{1}{2}]}{\Gamma[\frac{1}{2} + i\frac{m^2}{2qE}]}\right\} \qquad . \tag{6.27}$$

In the same way, we obtain:

$$
\begin{aligned}
\varpi_\omega^{p\,out} &= ie^{i\psi'}\sqrt{q_1}\,\varpi_\omega^{p\,in} + e^{-\frac{\pi m^2}{2qE}}e^{\frac{\pi\omega}{a}}\sqrt{q_1}e^{i\psi'}\Omega_\omega^{p\,in} + ie^{-\frac{\pi m^2}{2qE}}\Omega_\omega^{a\,in\,*} & , \\
\Omega_\omega^{a\,out} &= ie^{i\psi'}\sqrt{q_1}\,\Omega_\omega^{a\,in} + ie^{-\frac{\pi m^2}{2qE}}\varpi_\omega^{p\,in\,*} + e^{i\psi'}e^{-\frac{\pi m^2}{2qE}}e^{\frac{\pi\omega}{a}}\sqrt{q_1}\varpi_\omega^{a\,in} & , \\
\varpi_\omega^{a\,out} &= ie^{i\psi'}\sqrt{q_1}\,\varpi_\omega^{a\,in} + e^{i\psi'}e^{-\frac{\pi m^2}{2qE}}e^{\frac{\pi\omega}{a}}\sqrt{q_1}\Omega_\omega^{a\,in} + ie^{-\frac{\pi m^2}{2qE}}\Omega_\omega^{p\,in} & .
\end{aligned}
\tag{6.28}
$$

We found instructive to recover, using these basis, the decay rate (3.13) between *in* and *out* Minkowskian vacua. Formally we obtain :

$$|<0,out|0,in>|^2 = \exp\left[-2\frac{\mathcal{T}}{2\pi}\int_{-\infty}^{\infty}d\omega\,\ln\left(1 + e^{-\frac{\pi m^2}{qE}}\right)\right] \qquad . \tag{6.29}$$

Let us emphasize that $2(\mathcal{T}/2\pi)\int_{-\infty}^{\infty}d\omega = 2(\sum_{\omega>0} + \sum_{\omega<0})$ gives the expected total space-time volume factor $LT/2\pi$, in accordance with our interpretation of $\sum_{\omega>0}$ and $\sum_{\omega<0}$ as the space-time volume of the various quadrant.

More interesting is the computation of Rindlerian particle population in Minkowski vacua. Expressing the field as superposition of Unruh modes and Rindler we obtain, thanks to eqs (6.13-6.16), we obtain the Bogoljubov transformation between Unruh and Rindler creation and annihilation operators :

$$
\begin{aligned}
a_{\mathcal{U}_R}^{in}(\omega) &= \alpha_{\Omega\mathcal{U}_R}^{in}(\omega)a_\Omega^{in}(\omega) + \gamma_{\Omega\mathcal{U}_R}^{in}(\omega)b_\Omega^{in\,\dagger}(\omega) & \text{with } \omega > 0 & , \\
a_{\mathcal{V}_L}^{in}(\omega) &= \alpha_{\Omega\mathcal{V}_L}^{in}(\omega)a_\Omega^{in}(\omega) + \gamma_{\Omega\mathcal{V}_L}^{in}(\omega)b_\Omega^{in\,\dagger}(\omega) & \text{with } \omega < 0 & , \\
b_{\mathcal{U}_R}^{in\,\dagger}(\omega) &= \beta_{\Omega\mathcal{U}_R}^{in}(\omega)a_\Omega^{in}(\omega) + \epsilon_{\Omega\mathcal{U}_R}^{in}(\omega)b_\Omega^{in\,\dagger}(\omega) & \text{with } \omega < 0 & , \\
b_{\mathcal{V}_L}^{in\,\dagger}(\omega) &= \beta_{\Omega\mathcal{V}_L}^{in}(\omega)a_\Omega^{in}(\omega) + \epsilon_{\Omega\mathcal{V}_L}^{in}(\omega)b_\Omega^{in\,\dagger}(\omega) & \text{with } \omega > 0 & , \\
a_{\mathcal{V}_R}^{in}(\omega) &= \alpha_{\varpi\mathcal{V}_R}^{in}(\omega)a_\varpi^{in}(\omega) & , \\
b_{\mathcal{U}_L}^{in\,\dagger}(\omega) &= \epsilon_{\varpi\mathcal{U}_L}^{in}(\omega)b_\varpi^{in\,\dagger}(\omega) & ,
\end{aligned}
\tag{6.30}
$$

the various coefficient α, β, γ etc . . . being defined implicitly by eqs (6.6, 6.14-6.16). These relations allows us to express [30] the *in* Minkowski vacuum as (see Appendix **B**):

$$
\begin{aligned}
|0, Mink, in\rangle = &\prod_{\omega>0}\frac{1}{|\alpha_{\Omega\mathcal{U}_R}^{in\,*}|^2}\prod_{\omega>0}\frac{1}{|\alpha_{\Omega\mathcal{V}_L}^{in\,*}|^2}\exp\left[\sum_{\omega>0}\frac{\beta_{\Omega\mathcal{V}_L}^{in\,*}}{\alpha_{\Omega\mathcal{U}_R}^{in\,*}}a_{\mathcal{U}_R}^{in\,\dagger}(\omega)b_{\mathcal{V}_L}^{in\,\dagger}(\omega)\right] \\
&\exp\left[\sum_{\omega<0}\frac{\beta_{\Omega\mathcal{U}_R}^{in\,*}}{\alpha_{\Omega\mathcal{V}_L}^{in\,*}}a_{\mathcal{V}_L}^{in\,\dagger}(\omega)b_{\mathcal{U}_R}^{in\,\dagger}(\omega)\right]|0\rangle_{in}^{\mathbf{R}} \otimes |0\rangle_{in}^{\mathbf{L}}
\end{aligned}
$$
$$\tag{6.31}$$

and can then be interpreted as a superposition of Rindler's pairs consisting in Rindlerian particles and antiparticles and their partners, anti particles and particles localized in the opposite

33

sector. Using the explicit form of the Bogoljubov coefficients given in Appendix **B**, we obtain the mean density number of Rindlerian *out* particles that an accelerated observer (in the sector **R**) detects when the quantum state is the *in* Minkowski vacuum :

$$
\begin{aligned}
n_{\mathcal{U}_{\omega R}^{out}} &= \langle 0, Mink, in | a_{\mathcal{U}_R}^{\dagger\, out}(\omega) a_{\mathcal{U}_R}^{out}(\omega) | 0, Mink\, in \rangle \\
&= \theta(\omega) |\alpha_{\mathcal{U}\mathcal{U}}^R|^2 |\gamma_{\Omega\mathcal{U}_R}^{in}(\omega)|^2 + \theta(-\omega)|\beta_{\mathcal{U}\mathcal{U}}^R|^2 |\epsilon_{\Omega\mathcal{U}_R}^{in}(\omega)|^2 \\
&= e^{-\frac{\pi m^2}{qE}}
\end{aligned}
\tag{6.32}
$$

and:

$$
\begin{aligned}
n_{\mathcal{V}_{\omega>0\,R}^{out}} &= \langle 0, Mink, in | a_{\mathcal{V}_R}^{\dagger\, out}(\omega) a_{\mathcal{V}_R}^{out}(\omega) | 0, Mink\, in \rangle \\
&= |\alpha_{\mathcal{V}\mathcal{U}}^R|^2 |\gamma_{\Omega\mathcal{U}_R}^{in}(\omega)|^2 \\
&= \frac{(1+e^{-\frac{\pi m^2}{qE}})}{e^{2\pi\frac{\omega}{a}}-1}
\end{aligned}
\tag{6.33}
$$

for the various particles, while the density number of antiparticles is :

$$
\begin{aligned}
n_{\mathcal{V}_{\omega<0\,R}^{out}} &= \langle 0, Mink, in | b_{\mathcal{V}_R}^{\dagger\, out}(\omega) b_{\mathcal{V}_R}^{out}(\omega) | 0, Mink, , in \rangle \\
&= |\epsilon_{\mathcal{V}\mathcal{U}}^R|^2 |\beta_{\Omega\mathcal{U}_R}^{in}(\omega)|^2 + |\gamma_{\mathcal{V}\mathcal{V}}^R|^2 |\alpha_{\varpi\mathcal{V}_R}^{in}(\omega)|^2 \\
&= \frac{(1+e^{-\frac{\pi m^2}{qE}}e^{-2\pi\frac{\omega}{a}})}{e^{-2\pi\frac{\omega}{a}}-1}
\end{aligned}
\tag{6.34}
$$

In the same way, an accelerated observer in the **L** sector will detect the following populations:

$$
n_{\mathcal{U}_{\omega>0\,L}^{out}} = n_{\mathcal{V}_{\omega>0\,R}^{out}} = \frac{1+e^{-\frac{\pi m^2}{qE}}}{e^{2\pi\frac{\omega}{a}}-1}
\quad,
\tag{6.35}
$$

$$
n_{\mathcal{V}_{\omega\,L}^{out}} = n_{\mathcal{U}_{\omega\,R}^{out}} = e^{-\frac{\pi m^2}{qE}}
\tag{6.36}
$$

of antiparticles and:

$$
n_{\mathcal{U}_{\omega<0\,L}^{out}} = n_{\mathcal{V}_{\omega<0\,R}^{out}} = \frac{(1+e^{-\frac{\pi m^2}{qE}}e^{-2\pi\frac{\omega}{a}})}{e^{-2\pi\frac{\omega}{a}}-1}
\quad.
\tag{6.37}
$$

of particles.

All these populations are given by Bose factor and corrective terms proportional to the Schwinger factor. On the other hand, an accelerated observer in the right sector measures a total charge given by:

$$
\begin{aligned}
Q_R &= \int_0^\infty d\omega\, N_{\mathcal{V}_{\omega\,R}^{out}} + \int_{-\infty}^\infty d\omega\, N_{\mathcal{U}_{\omega\,R}^{out}} - \int_{-\infty}^0 d\omega\, N_{\mathcal{V}_{\omega\,R}^{out}} \\
&= \int_0^\infty d\omega\, e^{-\pi\frac{m^2}{qE}}
\quad.
\end{aligned}
\tag{6.38}
$$

Charge conservation implies that a left observer sees:

$$
Q_L = -\int_0^\infty d\omega\, e^{-\pi\frac{m^2}{qE}}
\quad.
\tag{6.39}
$$

34

Note (a check of all this algebra) that, always due to charge conservation, the same expressions of Q_R and Q_L are obtained if we compute them using *in* density number of particles, but the individual contributions of $\mathcal{U}$ and $\mathcal{V}$ type of (anti)particles are different.

Of course the total charges Q_F and Q_P are zero.

Moreover, when $\frac{qE}{m^2} \to 0$ (i.e. for a weak electric field or massive particles), some populations (6.32, 6.36) vanish, and the other (6.33,6.34, 6.35,6.37) become Boltzmanian in character $\approx$ $1/e^{\frac{2\pi|\omega|}{a}} - 1$. This can be understood as follows. Let us consider, for instance, the $\mathcal{U}^{out}_{\omega R}$ modes. In the limit where qE goes to zero, these modes become localized near i^0_R. Their charges (see eqs (4.22)) on the horizon components vanishes, while on the past and future horizon they become equal to unity. Indeed these modes tend to Bessel-I functions, i.e. function which grows exponentially when ρ goes to infinity, and that do not contribute to the Hilbert space of the neutral quantum field (see for instance ref. [31]). Classically, they correspond to hyperbolic trajectories pushed away to infinity, hyperbolic trajectories whose radius (m/qE) become infinite and whose asymptotic points on $\mathcal{I}^-_R$ and $\mathcal{I}^+_R$ slip to i^0. On the contrary, $\mathcal{V}^{out}_{\omega R}$ correspond classically to hyperbolic motion whose past and future asymptotic points tend respectively near i^- and i^+ i.e. inertial trajectories. When such trajectories cross the **R** quadrant, they enter in and leave it out across the horizon components $\mathcal{H}^-_R$ and $\mathcal{H}^+_R$. This is confirmed by the fact that for such modes their charge contents (eqs (4.19)) vanishes at infinity but become equal to unity on the horizon components.

7   Uniformly accelerated charged detector

In a previous work [7] we have computed transition amplitudes (see hereafter, formulas 7.26 to 7.34) between charged particles of masses and charges that we denote here M, Q and M', Q' (instead of M, Q and m, q as in ref.[7]) interacting by exchange of a third kind of particles of mass m and charge q (that was called μ and α in ref.[7]). Their interactions were described by a three field interaction Hamiltonian and first order perturbation calculations were performed. Our purpose now is first to clarify some aspect of the physics of the accelerated detector and to make an explicit contact between it and the model built on the three interacting fields.

The detector can be seen as a "two-level ion" propagating in a constant electric field. The ion levels are supposed to have rest masses M and M' (resp. charges Q and Q'), both much greater than the mass m (resp. charge q) of the exchanged quanta. This ion makes transition, without recoil, between its two levels, i.e. whatever is its internal configuration, it always moves with constant acceleration $a = Q\,E/M = Q'\,E/M'$. Hereafter we show how this model appears as a limiting case of the three field model, by comparing amplitude of transition obtained in both cases. In the following, all the calculations are done at first order of the interaction coupling constant. The physics of the accelerated two level detector is described through the effective hamiltonian :

$$\mathcal{H}^{int}(\tau) = a\tilde{g} \left[A e^{-i\Delta M \tau} e^{iq \int A_\mu dx^\mu} \hat{\phi}^\dagger(\tau, \xi = 0) + A^\dagger e^{i\Delta M \tau} e^{-iq \int A_\mu dx^\mu} \hat{\phi}(\tau, \xi = 0) \right], \qquad (7.1)$$

where we have taken into account the prescribed trajectory of the detector $\xi = 0$, $(\rho = a^{-1})$. The operators A and $A^\dagger$ are annihilation and creation operator acting on the two dimensional

Hilbert space generated by the two quantum states *up* and *down* of the ion :

$$A|up\rangle = |down\rangle \quad , \quad A|down\rangle = 0 \quad , \quad A^\dagger|down\rangle = |up\rangle \quad , \quad A^\dagger|up\rangle = 0 \quad . \tag{7.2}$$

The energy difference between these two levels is given by the mass gap $\Delta M > 0$. Indeed, for the uniformly accelerated detector following the trajectory $\xi = 0$, the local rest frame time (the detector proper time) coincides with the Rindler τ time, fixing the conjugate energy to be the rest frame energy difference between the two levels. The integral phase factor $\int A_\mu \, dx^\mu$ insures gauge invariance of the interaction. It has to be computed along the trajectory of the detector, from an arbitrary origin to the position of the detector at interaction proper time τ. For the model we consider here, it reduces to $\frac{qE\tau}{2a}$, up to an arbitrary constant of integration.

Before discussing this model, we would like to recall the reader a few results [6] concerning the simpler situation where the exchanged agent is massless and chargeless. In this case the field can be expressed, on **R**, as a superposition of Unruh U and V modes (see ref. [8] for details) :

$$\hat{\phi}_R = \hat{\phi}_R^U + \hat{\phi}_R^V = \int_0^\infty d\omega \, \left(a_\omega^U \, \varphi_\omega^U + a_\omega^V \varphi_\omega^V \right) + \text{herm. conj.} \tag{7.3}$$

The detector transitions, when the field state is the vacuum, can only occur by emission of Minkowskian neutral quanta. At first order of perturbation, the probability amplitudes of such transitions from the lower mass state of the detector to the higher one (denoted by B) or the converse (denoted by A) via the emission of one quantum associated to an Unruh U mode are given by :

$$
\begin{aligned}
B(\omega; \Delta M) &= \langle O_M|\langle +|a_\omega \mathcal{T}_\tau e^{-i\int d\tau \mathcal{H}^i nt}|-\rangle|0_M\rangle &= -i\,\tilde{g}\sqrt{\tfrac{\pi}{|\omega|}}\alpha_\omega \delta(\tfrac{\Delta M}{a} + \tfrac{\omega}{a}) \\[2mm]
A(\omega; \Delta M) &= \langle O_M|\langle -|a_\omega \mathcal{T}_\tau e^{-i\int d\tau \mathcal{H}^i nt}|+\rangle|0_M\rangle &= -i\,\tilde{g}\sqrt{\tfrac{\pi}{|\omega|}}\alpha_\omega \delta(\tfrac{\Delta M}{a} - \tfrac{\omega}{a})
\end{aligned}
\tag{7.4}
$$

where $\alpha_\omega = 1/\sqrt{(1 - e^{-\frac{2\pi \omega}{a}})}$ and $\mathcal{T}_\tau$ denotes the time ordering operator with respect to the detector proper time. Its is interesting to compare these formula with their analog where the emitted quanta are those associated to Minkowski modes of energy k (eq. (2.48) of ref. [8]). In the latter case the instant of emission is given by $\tilde{\tau}(k) = a^{-1}\ln k/\Delta M$ with a width $1/\sqrt{\Delta M}$. When the emitted quanta are in pure Unruh's states, their "energy" are fixed ($\omega = -\Delta M$), but the process becomes completely delocalized in time. In other words the Rindler "energy" balance is actually a "boost" constant of motion conservation relation. When recoils effects are taken into account but the classical picture of the detector still valid, the ion jumps from one hyperbola to another whose centers are such that their difference in localization remains in accord with eq.(2.9) for the different values of ω. Summing the square of these amplitudes and interpreting as usual the $\delta(0)$, we obtain the detector excitation probability per unit of proper time :

$$PB(\Delta M) = \int d\omega |B(\omega; \Delta M)|^2 = \tilde{g}^2 \frac{\pi}{\Delta M} \frac{1}{(e^{\frac{2\pi \Delta M}{a}} - 1)} \tag{7.5}$$

and similarly for the desexcitation probability per unit of proper time :

$$PA(\Delta M) = \tilde{g}^2 \frac{\pi}{\Delta M} \frac{1}{(1 - e^{-\frac{2\pi \Delta M}{a}})} \tag{7.6}$$

The ratio of these probabilities is:

$$\frac{PB(\Delta M)}{PA(\Delta M)} = e^{-\frac{2\pi}{a}\Delta M} \tag{7.7}$$

which reflects the thermal equilibrium of the detector and the radiation, at Unruh temperature $a/2\pi$.

Now let us repeat the same calculation, but in the framework of a charged exchanged agent. First we compute the probability amplitude of spontaneous excitation of the detector by emission of an Ω (resp ϖ) antiparticle of quantum number ω. Expressing the quantum field $\hat{\phi}$ as a superposition of Unruh modes, we obtain at first order in the coupling constant :

$$B\binom{\Omega_\omega}{\varpi_\omega}; \Delta M) = -i\tilde{g}a\langle 0, Mink, out|b^{out}_{\{\substack{\Omega\\\varpi}}(\omega)\int\mathcal{H}^{int}(\tau)d\tau|0, Mink, in\rangle \tag{7.8}$$

$$. = -i\frac{\tilde{g}\,a}{N}\int d\tau e^{i\tau(\Delta M-\frac{qE}{2a})}\left\{\Omega^{a\,out\,*}_\omega - i\frac{\tilde{g}\,a}{NqE}\frac{\delta}{\gamma}\int d\tau e^{i\tau(\Delta M-\frac{qE}{2a})}\right.$$

$$\left.\int_{-\infty}^\infty d\sigma\int_{-\infty}^\infty d\tilde{\omega}\left\{\alpha^{out\,*}_{\Omega\varphi}(\sigma,\tilde{\omega})\epsilon^{out}_{\{\substack{\Omega\\\varpi}\,\varphi}(\sigma,\omega)\Omega^{p\,out}_{\tilde{\omega}} + \alpha^{out\,*}_{\varpi\varphi}(\sigma,\tilde{\omega})\epsilon^{out}_{\{\substack{\Omega\\\varpi}\,\varphi}(\sigma,\omega)\varpi^{p\,out}_{\tilde{\omega}}\right\} \tag{7.9}$$

Here $N =< 0, Mink, out|0, Mink, in >$ is the normalization factor (3.12), δ/γ is defined by eq; (3.12) and $\alpha^{out}_{\{\substack{\Omega\\\varpi}\,\varphi}$ and $\epsilon^{out}_{\{\substack{\Omega\\\varpi}\,\varphi}$ are the coefficients of the (non frequency mixing) Bogoljubov transformation between Unruh and Minkowski modes (6.19 6.22). Finally, expressing the Unruh modes in terms of Rindler modes (eqs B.12), the τ integration in eq. (7.9) becomes trivial and gives δ functions. So we obtain the excitation amplitude probability $B(\Omega_\omega; \Delta M)$ of the detector by emission of an Ω_ω quantum as the sum of

$$-i\frac{\tilde{g}\sqrt{2\pi}}{N}\epsilon^{out}_{\Omega\mathcal{V}_R}\mathcal{N}(\mathcal{V}^{out}_{\omega R})W_{-i(\frac{\omega}{2a}-\frac{m^2}{2qE}),-i\frac{\omega}{2a}}\left[\frac{iqE}{2a^2}\right]\delta(\frac{\Delta M}{a}-\frac{qE}{2a^2}-\frac{\omega}{a}). \tag{7.10}$$

and the integral :

$$-i\frac{\tilde{g}\sqrt{2\pi}}{NqE}\frac{\delta}{\gamma}\int_{-\infty}^\infty d\tilde{\omega}\int_{-\infty}^\infty d\sigma\,\delta(\frac{\Delta M}{a}-\frac{qE}{2a^2}-\frac{\tilde{\omega}}{a})$$

$$\left\{\alpha^{out\,*}_{\Omega\varphi}(\sigma,\tilde{\omega})\epsilon^{out}_{\Omega\varphi}(\sigma,\omega)\alpha^{out}_{\Omega\mathcal{V}_R}(\tilde{\omega})\mathcal{N}(\mathcal{V}^{out}_{\tilde{\omega},R})W_{-i(\frac{\tilde{\omega}}{2a}-\frac{m^2}{2qE}),-i\frac{\tilde{\omega}}{2a}}\left[\frac{iqE}{2a^2}\right]+\right.$$

$$\left.\alpha^{out\,*}_{\varpi\varphi}(\sigma,\tilde{\omega})\epsilon^{out}_{\Omega\varphi}(\sigma,\omega)\alpha^{out}_{\varpi\mathcal{U}_R}(\tilde{\omega})\mathcal{N}(\mathcal{U}^{out}_{\tilde{\omega} R})M_{i(\frac{\tilde{\omega}}{2a}-\frac{m^2}{2qE}),i\frac{\tilde{\omega}}{2a}}\left[-\frac{iqE}{2a^2}\right]\right\} \tag{7.11}$$

Using the relations (6.18) and (6.25) for integrals of products of $A(\sigma,\omega)$ functions, it easy to show that the σ integral gives a $\delta(\omega-\tilde{\omega})$ function; as a consequence the integral over $\tilde{\omega}$ is trivial to carry out. So we get for the second term in the expression (7.9) of $B(\Omega_\omega; \Delta M)$:

$$-i\frac{\tilde{g}\sqrt{2\pi}}{N}\frac{\delta}{\gamma}\delta(\frac{\Delta M}{a}-\frac{qE}{2a^2}-\frac{\omega}{a})\Gamma(\frac{1}{2}-i\frac{\omega}{a}+i\frac{m^2}{2qE})\frac{1}{\sqrt{2\pi}}$$

$$\left\{e^{i\frac{\pi}{2}}e^{\frac{\pi\omega}{2a}}e^{-\frac{\pi m^2}{4qE}}\alpha^{out}_{\Omega\mathcal{V}_R}(\omega)\mathcal{N}(\mathcal{V}^{out}_{\omega,R})W_{-i(\frac{\omega}{2a}-\frac{m^2}{2qE}),-i\frac{\omega}{2a}}\left[\frac{iqE}{2a^2}\right]+\right.$$

$$\left.e^{-i\psi}e^{-\frac{\pi\omega}{2a}}e^{\frac{\pi m^2}{4qE}}\alpha^{out}_{\varpi\mathcal{U}_R}(\omega)\mathcal{N}(\mathcal{U}^{out}_{\omega R})M_{i(\frac{\omega}{2a}-\frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-\frac{iqE}{2a^2}\right]\right\} ,$$

$$\tag{7.12}$$

which, once the explicit expressions of the coefficients $\epsilon^{out}_{\Omega \mathcal{V}_R}$ and $\alpha^{out}_{\Omega \mathcal{V}_R}$ (eqs (B.13)) are explicited, lead to:

$$
\begin{aligned}
B(\Omega_\omega; \Delta M) &= -i\frac{\tilde{g}\sqrt{2\pi}}{2N}\delta(\frac{\Delta M}{a} - \frac{qE}{2a^2} - \frac{\omega}{a})(\frac{qE}{a})^{-\frac{1}{2}}\frac{e^{-\frac{3\pi\omega}{4a}}e^{\frac{\pi m^2}{4qE}}}{\cosh \pi(\frac{\omega}{a} - \frac{m^2}{2qE})} \\
&\quad \frac{\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})}{\Gamma(\frac{1}{2} + i\frac{\omega}{a} - i\frac{m^2}{2qE})}W_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-\frac{iqE}{2a^2}\right].
\end{aligned}
\tag{7.13}
$$

In the same way we obtain the excitation amplitude via emission of a ϖ_ω antiparticle :

$$
\begin{aligned}
B(\varpi_\omega; \Delta M) &= -i\frac{\tilde{g}\sqrt{2\pi}}{NqE}\frac{\delta}{\gamma}\int_{-\infty}^{\infty}d\tilde{\omega}\int_{-\infty}^{\infty}d\sigma\,\delta(\frac{\Delta M}{a} - \frac{qE}{2a^2} - \frac{\tilde{\omega}}{a}) \\
&\quad \left\{ \alpha^{out\,*}_{\Omega\varphi}(\sigma,\tilde{\omega})\epsilon^{out}_{\varpi\varphi}(\sigma,\omega)\alpha^{out}_{\Omega\mathcal{V}_R}(\tilde{\omega})\mathcal{N}(\mathcal{V}^{out}_{\tilde{\omega},R})W_{-i(\frac{\text{\o}mega}{2a} - \frac{m^2}{2qE}),-i\frac{\tilde{\omega}}{2a}}\left[\frac{iqE}{2a^2}\right] + \right. \\
&\quad \left. \alpha^{out\,*}_{\varpi\varphi}(\sigma,\tilde{\omega})\epsilon^{out}_{\varpi\varphi}(\sigma,\omega)\alpha^{out}_{\varpi\mathcal{U}_R}(\tilde{\omega})\mathcal{N}(\mathcal{U}^{out}_{\tilde{\omega}\,R})M_{i(\frac{\tilde{\omega}}{2a} - \frac{m^2}{2qE}),i\frac{\tilde{\omega}}{2a}}\left[-\frac{iqE}{2a^2}\right] \right\}
\end{aligned}
\tag{7.14}
$$

which, again using the relations (6.18, 6.25), lead to :

$$
\begin{aligned}
B(\varpi_\omega; \Delta M) &= -i\frac{\tilde{g}\sqrt{2\pi}}{2N}\delta(\frac{\Delta M}{a} - \frac{qE}{2a^2} - \frac{\omega}{a})(\frac{qE}{a})^{-\frac{1}{2}}e^{-i\psi} \\
&\quad \frac{e^{i\frac{\pi}{2}}e^{\frac{\pi\omega}{4a}}e^{-\frac{\pi m^2}{4qE}}}{\cosh \pi(\frac{\omega}{a} - \frac{m^2}{2qE})}\frac{\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})}{\Gamma(\frac{1}{2} - i\frac{m^2}{2qE} + i\frac{\omega}{a})}W_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-\frac{iqE}{2a^2}\right].
\end{aligned}
\tag{7.15}
$$

Similar calculations lead the desexcitation amplitude of probability of the detector, by emission of a particle of type Ω or ϖ :

$$
\begin{aligned}
A(\Omega_\omega; \Delta M) &= -i\frac{\tilde{g}\sqrt{2\pi}}{2N}\delta(\frac{\Delta M}{a} - \frac{qE}{2a^2} + \frac{\omega}{a})(\frac{qE}{a})^{-\frac{1}{2}}\frac{\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})}{\Gamma(\frac{1}{2} + i\frac{\omega}{a} - i\frac{m^2}{2qE})} \\
&\quad \frac{e^{\frac{\pi\omega}{4a}}e^{\frac{\pi m^2}{4qE}}}{\cosh \pi(\frac{\omega}{a} - \frac{m^2}{2qE})}W_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-\frac{iqE}{2a^2}\right],
\end{aligned}
\tag{7.16}
$$

and:

$$
\begin{aligned}
A(\varpi_\omega; \Delta M) &= -i\frac{\tilde{g}\sqrt{2\pi}}{N}\delta(\frac{\Delta M}{a} - \frac{qE}{2a^2} + \frac{\omega}{a})(\frac{qE}{a})^{-\frac{1}{2}}\left(\frac{|\sinh\frac{\pi\omega}{a}|}{\cosh \pi(\frac{\omega}{a} - \frac{m^2}{2qE})}\right)^{\frac{1}{2}}\varepsilon(\omega)|\frac{\omega}{a}|^{-\frac{1}{2}}\Gamma[1 - i\frac{\omega}{a}] \\
&\quad \left\{\frac{1}{\Gamma(\frac{1}{2} - i\frac{m^2}{2qE})}\frac{e^{\frac{3\pi m^2}{4qE}}e^{-\frac{3\pi\omega}{4a}}}{2\cosh\frac{\pi m^2}{2qE}}W_{i(\frac{\omega}{2a} - \frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-\frac{iqE}{2a^2}\right] + \frac{1}{\Gamma(\frac{1}{2} - i\frac{\omega}{a} + i\frac{m^2}{2qE})}e^{i\frac{\pi}{2}}e^{\frac{\pi m^2}{4qE}}e^{-\frac{\pi\omega}{4a}}W_{-i(\frac{\omega}{2a} - \frac{m^2}{2qE}),-i\frac{\omega}{2a}}\left[\frac{iqE}{2a^2}\right]\right\} .
\end{aligned}
$$

On the other hand let us notice that the transition amplitudes due to absorption of quanta :

$$
B'(\frac{\Omega_\omega}{\varpi_\omega}; \Delta M) = \langle +|\langle 0, Mink, out|\mathcal{T}e^{-i\int d\tau\,\mathcal{H}^{int}(\tau)}a^{in\,\dagger}_{\{\substack{\Omega\\\varpi}}(\omega)|0, Mink, in\rangle|-\rangle
$$

38

$$(7.17)$$

$$A'(\frac{\Omega_\omega}{\varpi_\omega}; \Delta M) = \langle -|\langle 0, Mink, out| \mathcal{T} e^{-i \int d\tau \, \mathcal{H}^{int}(\tau)} b_{\{\frac{\Omega}{\varpi}}^{in\,\dagger}(\omega)|0, Mink, in\rangle|+\rangle$$

$$(7.18)$$

are directly related to the previous ones by :

$$B'(\frac{\Omega_\omega}{\varpi_\omega}; \Delta M) = A(\frac{\Omega_\omega}{\varpi_\omega}; \Delta M) \qquad \text{and} \qquad A'(\frac{\Omega_\omega}{\varpi_\omega}; \Delta M) = (\frac{\Omega_\omega}{\varpi_\omega}; \Delta M) \qquad . \qquad (7.19)$$

These last relation can be verified by repeating the previous calculation, or more directly using **CPT** transformation arguments [6].

The occurrence of the Dirac distribution $\delta(\frac{\Delta M}{a} - \frac{qE}{2a^2} + \frac{\omega}{a})$ in all these amplitudes makes trivial the estimation of probabilities of transition per unit of proper time. With obvious notation, T beeing a large detector proper time interval, we obtain :

$$PB(\Omega, \Delta M) = T^{-1} \int |B(\Omega_\omega; \Delta M)|^2 \, d\omega = |\mathcal{C}|^2 \left| e^{-\frac{3\pi\omega_\star}{2a}} W_{i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), i\frac{\omega_\star}{2a}} \left[-\frac{iqE}{2a^2} \right] \right|^2 \qquad (7.20)$$

$$PB(\varpi, \Delta M) = |\mathcal{C}|^2 e^{\frac{\pi\omega_\star}{2a}} e^{-\pi\frac{m^2}{qE}} \left| W_{i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), i\frac{\omega_\star}{2a}} \left[-\frac{iqE}{2a^2} \right] \right|^2 \qquad (7.21)$$

$$PA(\Omega, \Delta M) = |\mathcal{C}|^2 e^{\frac{\pi\omega_\star}{2a}} \left| W_{i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), i\frac{\omega_\star}{2a}} \left[-\frac{iqE}{2a^2} \right] \right|^2 \qquad (7.22)$$

$$PA(\varpi, \Delta M) = |\mathcal{C}|^2 e^{-\frac{\pi\omega_\star}{2a}} \frac{\pi}{\cosh \pi \frac{m^2}{2qE}} \left| W_{i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), i\frac{\omega_\star}{2a}} \left[-\frac{iqE}{2a^2} \right] \right|^2$$

$$\left| \frac{1}{\Gamma(1 - i\frac{\omega_\star}{a})} \frac{M_{-i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), -i\frac{\omega_\star}{2a}} \left[\frac{iqE}{2a^2} \right]}{W_{i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), i\frac{\omega_\star}{2a}} \left[-\frac{iqE}{2a^2} \right]} + \frac{e^{-i\frac{\pi}{2}} e^{-\pi\frac{m^2}{2qE}}}{\Gamma(\frac{1}{2} - i\frac{\omega_\star}{a} + i\frac{m^2}{2qE})} \frac{W_{-i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), -i\frac{\omega_\star}{2a}} \left[\frac{iqE}{2a^2} \right]}{W_{i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), i\frac{\omega_\star}{2a}} \left[-\frac{iqE}{2a^2} \right]} \right|^2$$

$$(7.23)$$

with $\omega_\star \equiv \Delta M - qE/2a$ fixed by the mass shell condition dictated by the delta functions and where we have set $|\mathcal{C}|^2 = \tilde{g}^2 \frac{\pi}{2N^2} (\frac{qE}{a})^{-1} e^{\pi\frac{m^2}{2qE}} / \left(\cosh \pi(\frac{\omega_\star}{a} - \frac{m^2}{2qE}) \cosh \pi \frac{m^2}{2qE} \right)$

In the limit of small charge exchange ($\frac{qE}{m^2} \mapsto 0$), approximating, in the limit ($qE \mapsto 0$), Whittaker functions by Bessel functions (see ref. [32], section **13.3**) :

$$\Gamma(\frac{1}{2} + i\frac{\omega_\star}{a} - i\frac{m^2}{2qE}) W_{-i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), -i\frac{\omega_\star}{2a}} \left[\frac{iqE}{2a^2} \right] \approx 2(\frac{m}{2a})^{i\frac{\omega_\star}{a}} (\frac{iqE}{2a^2})^{\frac{1}{2} - i\frac{\omega_\star}{2a}} K_{-i\frac{\omega_\star}{a}}(\frac{m}{a})$$

$$\frac{1}{\Gamma(\frac{1}{2} - i\frac{\omega_\star}{a})} M_{-i(\frac{\omega_\star}{2a} - \frac{m^2}{2qE}), -i\frac{\omega_\star}{2a}} \left[\frac{iqE}{2a^2} \right] \approx (\frac{m}{2a})^{i\frac{\omega_\star}{a}} (\frac{iqE}{2a^2})^{\frac{1}{2} - i\frac{\omega_\star}{2a}} I_{-i\frac{\omega_\star}{a}}(\frac{m}{a})$$

it is easy to see that the probabilities $PA(\Omega, \Delta M)$ and $PB(\Omega, \Delta M)$ are of order one compared to the probabilities $PA(\varpi, \Delta M)$ and $PB(\varpi, \Delta M)$ which of order $e^{-\pi m^2/qE}$. Indeed an amplitude such as $B(\varpi_\omega; \Delta M)$ corresponds to the probability amplitude of excitation of the detector by absorption of a quantum described by a ϖ_ω mode, i.e. a rindlerian mode $\mathcal{U}_R^{out}$ on the **R** quadrant. But, as discussed at the end of the previous section (see eq. (6.32)), such modes corresponds to fluctuations produced by the Schwinger mechanism, and in the limit of vanishing

charge, their populations become exponentially dampen (here we suppose that $qE/m^2 << 1$) compared to the populations of fluctuations due to the Unruh mechanism. Accordingly, the probabilities of transition involving such modes are also exponentially small, compared to those refereeing to process involving the modes created by the Unruh mechanism. This illustrate how the two kinds of creation process cooperate to the physics of the detector. This situation is similar to the physics of the Reissner-Nordström black-holes which start to loose their charge by the Schwinger mechanism, and after by Hawking evaporation [33]. In the limit of small charges we obtain as ratio of the probabilities of transition the thermodynamical equilibrium expression :

$$\frac{PB(\Delta M)}{PA(\Delta M)} = \frac{[PB(\Omega; \Delta M) + PB(\varpi; \Delta M)]}{[PA(\Omega; \Delta M) + PA(\varpi; \Delta M)]} \approx e^{-\frac{2\pi}{a}(\Delta M - \frac{qE}{2a})} \tag{7.24}$$

whose meaning is discussed at length in [7]. On fig. (**5**) we have plot this ratio of total probabilities divided by the Boltzmanian factor $\exp[-(2\pi(\Delta M/a - qe/2a^2) \equiv \exp[-2\pi\omega_\star]$. This figure illustrate the role played by the Schwinger factor which control the instability of the quantum state of exchanged quanta. For large mass or, equivalently small charge, we recover the usual Boltzman equilibrium formula, otherwise new equilibrium relations have to be considered. Once again the same physics occurs when we consider a quantum state in a strong gravitational field. For instance if a black hole is enclosed in a box, it may reach an equilibrium configuration with surrounding radiation. However the radiation alone will not obey a local equation of state linking energy density and pressure, but it is only the whole system that will satisfy an equilibrium relation involving its total mass energy, the geometry of the box, etc.

In ref [7] we also obtained expressions for the transition amplitudes of an accelerated ion, coupled to a massive and charged scalar field Φ_m of mass m and charge $q = Q - Q'$, when its recoil is taken into account. The two level ion was modeled by introducing two scalar fields Ψ_M and Ψ'_M, of masses M and M' and charges Q and Q'. The transitions were dictated through the interaction hamiltonian :

$$H^{int} = g \int dx \left(\Psi^\dagger_M \Psi_{M'} \Phi_m + \Psi_M \Psi^\dagger_{M'} \Phi^\dagger_m \right) \qquad . \tag{7.25}$$

We get for the desexcitation amplitude :

$$
\begin{aligned}
\mathcal{A}(k|k', k'') &\equiv \langle 0, out \mid a^{out}_m(k'') \, a^{out}_{M'}(k') \mathcal{T} e^{(-i)\int dt H^{int}} a^{in\dagger}_M(k) \mid 0, in \rangle \\
&= \mathrm{C}\, \delta(k - k' - k'') \mathbf{A}(k|k', k'')
\end{aligned}
\tag{7.26}
$$

which at first order of a perturbation expansion reduces to

$$
\begin{aligned}
\mathbf{A}(k|k', k'') &= \frac{\sqrt{2\pi}e^{-\frac{i\pi}{4}}e^{-\frac{\pi}{2}(\epsilon_{M'}+\epsilon_m)}e^{\frac{\pi\mathcal{E}}{4}}e^{\frac{i}{2}(\frac{k^2}{QE} - \frac{k'^2}{Q'E} - \frac{k''^2}{qE})}e^{\frac{i\mathcal{Q}^2}{2}}}{(2QE)^{1/4}(2Q'E)^{1/2}(2qE)^{1/2}\Gamma(i\epsilon_m + 1/2)\Gamma(i\epsilon_\mu + 1/2)} \\
&\qquad \Gamma(i\mathcal{E} + \frac{1}{2})(\frac{Q}{q})^{\frac{i}{2}\epsilon_\mu}(\frac{Q}{\alpha})^{\frac{i}{2}\epsilon_{M'}}\mathcal{I}
\end{aligned}
\tag{7.27}
$$

which $\mathcal{I}$ denoting the complicated expression:

$$\mathcal{I} = \left\{ \left(\frac{\sqrt{qE}}{\sqrt{Q'E}}\right)^{\frac{1}{2}+i\epsilon_M} D_{-i\mathcal{E}-\frac{1}{2}}\left[+\sqrt{2}e^{i\frac{\pi}{4}}\mathcal{Q}\right] B(i\epsilon_m + \frac{1}{2}, -i\mathcal{E} + \frac{1}{2})_2F_1(i\epsilon_M + \frac{1}{2}, i\epsilon_m + \frac{1}{2}, 1 + i\epsilon_M - i\epsilon_{M'}; -\frac{qE}{Q'E})\right.$$

$$
+ \left(\frac{\sqrt{Q'E}}{\sqrt{qE}}\right)^{\frac{1}{2}+i\epsilon_M} D_{-i\mathcal{E}-\frac{1}{2}}\left[\sqrt{2}e^{-i\frac{\pi}{4}}\mathcal{Q}\right] B(i\epsilon_{M'}+\frac{1}{2},-i\mathcal{E}+\frac{1}{2})\, {}_2F_1(i\epsilon_M+\frac{1}{2},i\epsilon_{M'}+\frac{1}{2},1+i\epsilon_M-i\epsilon_m;-\frac{Q'E}{qE})\Big\}
$$

$$(7.28)$$

and where we have set:

$$
\mathcal{Q} = \frac{Q'E\,k - QE\,k'}{\sqrt{QEQ'EqE}} \qquad , \tag{7.29}
$$

$$
\mathcal{E} = \epsilon_{M'} + \epsilon_m - \epsilon_M \tag{7.30}
$$

while the constant C is related to the normalization factors and Bogoljubov coefficients through :

$$
\mathrm{C} = \frac{-ig}{\sqrt{2\pi}} \frac{1}{\alpha_M \mathcal{N}_M \alpha_{M'} \mathcal{N}_{M'} \alpha_m \mathcal{N}_m} \qquad . \tag{7.31}
$$

Similarly, the amplitude of spontaneous excitation for the two level ion, defined as:

$$
\mathcal{B}(k'|k,-k'') = \langle 0, out \mid a_m^{out}(k'')\, b_M^{out}(-k)\, \tau e^{-i\int dt H^{int}} b_{M'}^{in\dagger}(-k') \mid 0, in\rangle \tag{7.32}
$$

was obtained into the form:

$$
\mathcal{B}(k'|k,-k'') = \mathrm{C}\,\delta(k-k'-k'')\mathbf{B}(k'|k,-k'') \tag{7.33}
$$

with at first order:

$$
\begin{aligned}
\mathbf{B}(k'|k,-k'') &= (\frac{q}{Q})^{\frac{i}{2}(\epsilon_m-i\epsilon_M)}(\frac{\alpha}{Q})^{\frac{i}{2}(\epsilon_{M'}-i\epsilon_M)} \frac{e^{-\frac{\pi}{2}(\epsilon_{M'}+\epsilon_m)}e^{\frac{i\pi}{4}}\sqrt{2\pi}e^{\frac{i}{2}(\frac{k^2}{QE}-\frac{k'^2}{Q'E}+\frac{k''^2}{qE})}}{(2QE)^{3/4}(2Q'E)^{1/4}(2qE)^{1/4}\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_{M'}+1/2)} \\
&\quad \Gamma(i\mathcal{E}+\frac{1}{2})e^{\frac{i\mathcal{Q}^2}{2}}e^{\frac{\pi\mathcal{E}}{8}}e^{-\frac{i\pi}{8}}D_{-i\mathcal{E}-\frac{1}{2}}\left[\sqrt{2}e^{i\frac{5\pi}{4}}\mathcal{Q}\right] \\
&\quad \Big\{ e^{-\frac{3i\pi}{8}}B(i\epsilon_m+\frac{1}{2},-i\epsilon_M+\frac{1}{2})\, {}_2F_1(i\mathcal{E}+\frac{1}{2},\frac{1}{2}-i\epsilon_M,1+i\epsilon_m-i\epsilon_M;\frac{Q'E}{QE}) \\
&\quad + e^{-\pi\epsilon_M}e^{\frac{i\pi}{8}}B(i\epsilon_{M'}+\frac{1}{2},-i\epsilon_M+\frac{1}{2})\, {}_2F_1(i\mathcal{E}+\frac{1}{2},\frac{1}{2}-i\epsilon_M,1+i\epsilon_{M'}-i\epsilon_M;\frac{qE}{QE})\Big\} \qquad .
\end{aligned}
$$

$$(7.34)$$

These amplitudes were obtained by using modes solving the filed equation by separation of variables in the gauge $A = -Etdz$, modes such that their spatial dependence are of the form e^{ikz}. To make contact between these amplitudes and those obtain in the framework of the detector model we first have to change our Fock space basis in order to use, in both situations, the same quantum numbers to label the states. These k modes, gauge transformed so as to solve eq. (3.7), are related to the modes Minkowski modes introduced in eq. (3.9) by the (trivial) Bogoljubov transformation:

$$
\begin{aligned}
\varphi_k^{p\,in} &= \frac{e^{i\frac{3\pi}{8}}}{\sqrt{2\pi Q'E}}\int d\sigma e^{-i\frac{\sigma k}{Q'E}}\varphi_\sigma^{p\,in} \qquad , \\
\varphi_{-k}^{a\,in*} &= \frac{e^{-i\frac{3\pi}{8}}}{\sqrt{2\pi Q'E}}\int d\sigma e^{-i\frac{\sigma k}{Q'E}}\varphi_\sigma^{a\,in*} \qquad , \\
\varphi_k^{p\,out} &= \frac{e^{i\frac{-3\pi}{8}}}{\sqrt{2\pi Q'E}}\int d\sigma e^{-i\frac{\sigma k}{Q'E}}\varphi_\sigma^{p\,out} \qquad , \\
\varphi_{-k}^{a\,out*} &= \frac{e^{i\frac{3\pi}{8}}}{\sqrt{2\pi Q'E}}\int d\sigma e^{-i\frac{\sigma k}{Q'E}}\varphi_\sigma^{a\,out*} \qquad .
\end{aligned}
$$

$$(7.35)$$

In the following, we only discuss the first order expansion of the $\mathcal{A}(k|k', k'')$ amplitude :

$$
\begin{aligned}
\mathcal{A}(k|k', k'') &= -i\langle 0, out \mid a_m^{out}(k'') \, a_{M'}^{out}(k') \int dt H^{int} a_M^{in\dagger}(k) \mid 0, in\rangle \\
&= -i \int d\sigma d\sigma' d\sigma'' a(\sigma'', -k'') a(\sigma', -k') a(\sigma, k)\langle 0, out \mid a_m^{out}(\sigma'') \, a_{M'}^{out}(\sigma') \int dt H^{int} a_M^{in\dagger}(\sigma) \mid 0, in\rangle
\end{aligned}
$$
(7.36)

and show how from this amplitude we may recover the transition amplitude of the two level detector.

We shall evaluate the limit of this "3-field" amplitudes of transition for $M, M' \to \infty$ with $\Delta M/M \equiv (M - M')/M \ll 1$ and $\Delta m/a \ll 1$, which corresponds to the situation where the three field model mimics the heavy two level detector, without recoil. The detector states are given by sharply localized states built out of the vacua $|0_M >$ and $|0_{M'} >$. They are obtained as :

$$
|+ >= \int f_+(k) dk a_M^\dagger(k)|0_M > |0_{M'} > \quad , \quad |- >= \int f_-(k') dk' a_{M'}^\dagger(k')|0_M > |0'_M > \quad (7.37)
$$

and the operator $A^\dagger$, acting on this two dimensional state space, is:

$$
A^\dagger = \int f_+(k) a_M^\dagger(k) \; dk \int f_-^*(k') a_{M'}^\dagger(k') dk' \tag{7.38}
$$

These states are supposed to be normed $(< +|+ >= \int f_+ f_+^* \, dk = 1, \; < -|- >= \int f_- f_-^* \, dk' = 1, \; < +|- >= 0)$ and, as in the large mass limit, the Schwinger process vanishes we have not to distinguish between *in* or *out* vacua $|0_M >$ and $|0_{M'} >$. At first order of perturbation, the amplitude of transition between these states becomes

$$
\begin{aligned}
\mathcal{A}(k'') &= \int f_+(k) dk \int f_-(k') dk' \mathcal{A}(k|k', k'') \tag{7.39} \\
&\simeq -ig < 0, m, out|a_m^{out}(k'') \int dt \, dz \, \phi_M(t, z) \hat{\Phi}_m^*(t, z) \phi_{M'}^*(t, z)|0, m, in > \tag{7.40}
\end{aligned}
$$

where $\mathcal{H}_{int}(\tau)$ is given by eq. (7.1) and where $\phi_M(t, z)$ and $\phi_{M'}(t, z)$ are classical solution of the wave equation, given by the superposition of modes weighted by the function $f_\pm(k)$; for example: $\phi_M(t, z) = \int f_+(k) \phi_k^p(t, z) \, dk$. Note that at this first order of the perturbation expansion, a necessary condition for $\mathcal{A}(k'')$ to be non zero is that the wave packets $\phi_M(t, z)$ and $\phi_{M'}(t, z)$ overlap, but at higher order virtual detector states allow "tunneling" transitions between non overlapping configurations of the detector. The heavy detector corresponds to the limit where these wave packets have their supports blend into a single classical trajectory, here a hyperbola whose center is at the origin of the (t, z) coordinates. Such configurations can be obtained by using, in the large mass limit, the maximally localized packets introduce in [9] or more directly just by considering suitable superpositions of W.K.B. approximate solutions of the wave equation :

$$
\phi_k^p(t, z) \approx \left(\frac{QEt}{M}\right)^{(i\frac{M^2}{2QE} - \frac{1}{2})} e^{-ik(z-t)} e^{i\frac{QEt(t-z)}{2}} \qquad . \tag{7.41}
$$

In the limit considered, we thus obtain:

$$\phi_M^*(t,z)\,\phi_{M'}(t,z) \;\;\propto\;\; e^{\frac{\Delta M}{2a}\ln at}\delta(\rho - a^{-1})$$

$$= \;\; e^{i\Delta M\,\tau}e^{-i\int A_\mu\,dx^\mu}\,\delta(\rho - a^{-1}) \tag{7.42}$$

because $\Delta M\,a = (Q - Q')E = qE$. So we recover, from the field description, the energy difference between the two detector levels as predicted by the "equivalence" principle. The amplitude reduces to :

$$\mathcal{A}(k'') \;\;\simeq\;\; -i\int d\sigma''a(\sigma'',-k'')\langle -|\langle 0,m,out|a_m^{out}(\sigma'')\int d\tau\mathcal{H}_{int}(\tau)|0,m,in\rangle|+\rangle \qquad ,\!(7.43)$$

and using the Bogoljubov transformation (B.12) between Unruh and Rindler modes, we get :

$$\mathcal{A}(k'') \;\;\simeq\;\; e^{-i\frac{\pi}{8}}\int \frac{e^{i\frac{\sigma k''}{qE}}}{\sqrt{2\pi qE}}\left\{\alpha_{\Omega\varphi}^{out}(\sigma,\omega)A(\Omega_\omega;\Delta M) + \alpha_{\varpi\varphi}^{out}(\sigma,\omega)A(\varpi_\omega;\Delta M)\right\}\frac{d\sigma}{\sqrt{qE}}\,d\omega \tag{7.44}$$

which, can be explicitly integrate (thanks to the integral representations of Whittaker and parabolic cylinder functions) to give at the end:

$$\mathcal{A}(k'') \;\;\simeq\;\; -\frac{g}{N}\frac{a}{(2qE)^{3/4}}e^{\frac{\pi m^2}{8qE}}\frac{\Gamma[\frac{1}{2} + i\frac{m^2}{2qE}]}{\cosh[\pi(\frac{m^2}{2Q'E} - \frac{\mu}{a})]}$$

$$\left\{D_{i\frac{\mu}{a} - i\frac{m^2}{2qE} - \frac{1}{2}}\left[-e^{i\frac{\pi}{4}}k''\sqrt{\frac{2}{qE}}\right]\frac{W_{-i(\frac{\mu}{2a} - \frac{m^2}{2qE}),-i\frac{\mu}{2a}}\left[\frac{iqE}{2a^2}\right]}{\Gamma[\frac{1}{2} + i\frac{m^2}{2qE}]} + \right.$$

$$\left. i\,D_{i\frac{\mu}{a} - i\frac{m^2}{2qE} - \frac{1}{2}}\left[e^{i\frac{\pi}{4}}k''\sqrt{\frac{2}{qE}}\right]e^{-\frac{\pi\mu}{2a}}\frac{M_{i(\frac{\mu}{2a} - \frac{m^2}{2qE}),i\frac{m}{2a}}\left[-\frac{iqE}{2a^2}\right]}{\Gamma[1 + i\frac{\mu}{a}]}\right\} \qquad , \tag{7.45}$$

where $\mu = \Delta\,M - Q'E/2a$.

This result can also directly be obtained from the expression (7.27) of the amplitude $\mathcal{A}(k|k',k'')$ by taking its limit for infinite masses and charges. Indeed, in this limit, the amplitude factorizes into a term involving only the k and k' momenta and a term that tends to eq. (7.45), the effective coupling constant $\tilde{g}$ being then implicitly defined as the product of g times a double integral involving the functions $f_+^*(k)$ and $f_-(k')$ and the k, k' term that comes out from the full amplitude.

First let us discuss the factor $\mathcal{I}$ (eq. 7.28). The limits of the parabolic functions D are simply obtained by direct substitution: $\mathcal{Q} \mapsto \frac{k''}{\sqrt{qE}}$ and $\mathcal{E} \mapsto -i\frac{\mathcal{Q}}{a} + i\frac{m^2}{2qE}$. For the hypergeometric functions occurring in $\mathcal{I}$, we use its series expansion and obtain :

$$\lim_{M,m\to\infty} {}_2F_1(i\epsilon_M + \frac{1}{2}, i\epsilon_m + \frac{1}{2}, 1 + i\epsilon_M - i\epsilon_{M'}; -\frac{qE}{Q'E}) = e^{-i\frac{qE}{4a^2}}(-i\frac{qE}{2a^2})^{-\frac{1}{2} - i\frac{\mathcal{Q}}{2a}}M_{i\frac{\mathcal{Q}}{2a} - i\frac{m^2}{2qE}, i\frac{\mathcal{Q}}{2a}}(-\frac{iqE}{2a^2}) \qquad , \tag{7.46}$$

while for the second, we first have to use the inversion relation to pass from the argument Q'/q to q/Q' and the Stirling formula to finally get:

$$\lim_{M,m\to\infty} {}_2F_1(i\epsilon_M + \frac{1}{2}, i\epsilon_m + \frac{1}{2}, 1 + i\epsilon_M - i\epsilon_m; -\frac{Q'E}{qE}) = \lim_{M,m\to\infty} e^{i\frac{\pi}{4}} e^{\pi\frac{m^2}{4qE}} \left(\frac{M^2}{2QE}\right)^{\frac{1}{2}-i\frac{m^2}{2qE}}$$

$$\left\{ \frac{\Gamma(-i\frac{\mathcal{Q}}{a})}{\Gamma(\frac{1}{2}-i\epsilon_m)} \left(\frac{Q'E}{qE}\right)^{-\frac{1}{2}-i\epsilon_M} e^{-\frac{\pi\mathcal{Q}}{2a}} \left(\frac{M^2}{2QE}\right)^{i\frac{\mathcal{Q}}{a}} e^{-i\frac{qE}{4a^2}} \left(-i\frac{qE}{2a^2}\right)^{-\frac{1}{2}-i\frac{\mathcal{Q}}{2a}} M_{i\frac{\mathcal{Q}}{2a}-i\frac{m^2}{2qE}, i\frac{\mathcal{Q}}{2a}}\left(-\frac{iqE}{2a^2}\right) \right.$$

$$\left. + \frac{\Gamma(i\frac{\mathcal{Q}}{a})}{\Gamma(\frac{1}{2}-i\epsilon_m+i\frac{\mathcal{Q}}{a})} \left(\frac{Q'E}{qE}\right)^{-\frac{1}{2}-i\epsilon_{M'}} e^{-i\frac{qE}{4a^2}} \left(-i\frac{qE}{2a^2}\right)^{-\frac{1}{2}+i\frac{\mathcal{Q}}{2a}} M_{i\frac{\mathcal{Q}}{2a}-i\frac{m^2}{2qE}, -i\frac{\mathcal{Q}}{2a}}\left(-\frac{iqE}{2a^2}\right) \right\} \tag{7.47}$$

Moreover this can further simplified using eqs (A.22) and the limit of the $\mathcal{A}$ amplitude reads:

$$\lim_{M,m\to\infty} \frac{-ig}{\sqrt{2\pi}} \frac{1}{\alpha_M \mathcal{N}_M \alpha_{M'}\mathcal{N}_{M'}\alpha_m\mathcal{N}_m} \frac{\sqrt{2\pi} e^{-\frac{i\pi}{4}} e^{-\frac{\pi}{2}(\epsilon_{M'}+\epsilon_m)} e^{\frac{\pi\mathcal{E}}{4}} e^{-\frac{i}{2}(\frac{k''^2}{qE})} e^{\frac{i\mathcal{Q}^2}{2}}}{(2QE)^{1/4}(2Q'E)^{1/2}(2qE)^{1/2}\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_m+1/2)}$$

$$\Gamma(\frac{1}{2} + i\frac{m^2}{2qE} - i\frac{\mathcal{Q}}{a})\left(\frac{Q}{q}\right)^{\frac{i}{2}\epsilon_m}\left(\frac{Q}{\alpha}\right)^{\frac{i}{2}\epsilon_{M'}}\mathcal{I}$$

$$= \lim_{M,m\to\infty} \frac{-ga}{2\pi}(\text{phase}) \frac{1}{\mathcal{N}_m} \frac{1}{(QE)^{1/4}(Q'E)^{3/4}(2qE)^{3/4}} \Gamma(\frac{1}{2} + i\frac{m^2}{2qE} - i\frac{\mathcal{Q}}{a})$$

$$\Gamma(\frac{1}{2} - i\frac{m^2}{2qE} + i\frac{\mathcal{Q}}{a})\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})e^{\frac{\pi m^2}{8qE}}$$

$$\left\{ \frac{1}{\Gamma(1+i\frac{\mathcal{Q}}{a})} e^{-\frac{\pi\mathcal{Q}}{2a}} e^{i\frac{\pi}{2}} D_{-i\mathcal{E}-\frac{1}{2}}\left[+\sqrt{2}e^{i\frac{\pi}{4}}\frac{k''}{\sqrt{qE}}\right] M_{i\frac{\mathcal{Q}}{2a}-i\frac{m^2}{2qE}, i\frac{\mathcal{Q}}{2a}}\left(-\frac{iqE}{2a^2}\right) \right.$$

$$\left. + \frac{1}{\Gamma(\frac{1}{2}+i\frac{m^2}{2qE})} D_{-i\mathcal{E}-\frac{1}{2}}\left[\sqrt{2}e^{-i\frac{\pi}{4}}\frac{k''}{\sqrt{qE}}\right] W_{-i\frac{\mathcal{Q}}{2a}+i\frac{m^2}{2qE}, i\frac{\mathcal{Q}}{2a}}\left(\frac{iqE}{2a^2}\right) \right\} \tag{7.48}$$

This is in perfect agreement with eq. (7.45) if the coupling constants $\tilde{g}$ and g are connected by a (q, m)-independent relation, dependent of the precise form of the weight functions characterizing the detector.

So we may relate the terms of the amplitude of transition (7.27) to Unruh modes, and we see that the physical reason that favor one kind of modes with respect to the other found its root in that their own existence is a manifestation of the Schwinger mechanism and that their probabilities of interaction is weighted by a Schwinger factor.

Conclusion

In this paper, we have studied, in Rindler coordinates, the quantization of a charged field interacting with a constant electric field. The main characteristic of this problem is that it involves two acceleration parameters: the acceleration of the Rindler observer (the detector of section **7**) and the natural acceleration of the charged quanta (qE/m). So we obtain a toy (but exactly solvable) model for the quantization of a charged field in a Reissner-Nordtröm black hole geometry, in the same way as the Unruh detector mimics the physics around a Schwarschild black hole. It is that similarity that constitutes the main motivation of our work.

Now, let us summarize the main points of our analysis:

- The quantization of a charged field in Rindler coordinates illustrates the "symmetry breaking" between particles and antiparticles in the **R** and **L** quadrants. Eqs (6.38) and (6.39) show that the Minkowskian vacuum state carries a positive charge on the **R** and (of course) the opposite charge on the **L** quadrant. This was expected, as antiparticles (*resp.* particles) are always obliged to leave the **R** (*resp.* **L**) quadrant, which is not necessarily the case for particles (*resp.* antiparticles) that may stay inside for ever. Nevertheless, although obvious, we find interesting to see how this property emerges from the rules of quantum mechanics and is encoded in the wave function of the various modes used. Let us note that when the quantization is performed on the full Minkowski space, no such charge polarization effect comes into evidence because in this framework operator expectation values are obtained by averaging over the complete space instead of over just one quadrant.

- The discussion of the classical trajectories that leads to interpret the conserved quantum number ω as the invariant distance Δ from the center of the hyperbolic trajectories to the common vertex of the four Rindler quadrants shows that (once more in terms of wave packets), it is near the center of the trajectories that the pair production mechanism occurs. Indeed Δ depends crucially on the sign of ω, the extra term $1/2$ being only the reflect of the quantum indeterminacy of the position. For instance, it is only for $\omega < 0$ (eqs (4.30, 4.31) that the Bogoljubov transformation is non trivial (i.e. mixes particles and antiparticles) on the **R** quadrant.

- On the Rindler quadrants **R** or **L** the persistence of the Rindler vacua is given by the usual Schwinger result modified by a surface term (4.33). The latter becomes negligible in the case of constant electric field in the large volume limit but it plays nevertheless an important rôle. In the framework of black hole physics it becomes the vacuum polarization term that cancels the Hawking radiation flux in the Boulware vacuum.

- The limitations of the W.K.B. evaluation of the Feynman propagator are exemplified by evaluating it on the one hand as a mode superposition and on the other hand using the Pauli-Van Vleck approximation of the Schwinger kernel. Here the Jacobi fields were an essential simplifying ingredient of the calculation, which shows that such approximation is nothing else than a quadratic expansion of the classical potential around its minimum (eqs 5.58, 5.59).

- The interpretation of the usual Minkowskian propagator as a sum over winding Rindlerian propagators is reinforced. We have shown that the rate of particle production calculated from the zero term of the Minkowskian propagator winding number expansion coincides with the rate obtained from the Rindler propagator.

- But on the other hand the calculation of the Rindlerian population of the Minkowski vacuum leads to mode densities that are not equilibrium distributions in character. This is not surprising as in a constant electric field there are no physical reasons to expect an equilibrium distribution.
  Moreover some of the distributions (6.32,6.36) obtained vanish when the electric field goes to zero. This reflects the fact that for a charged field we obtain twice the number of modes encountered when we quantize a neutral field. Indeed for the charged field the two

linearly independent solutions of the radial equation (4.2) have to be taken into account while for the uncharged field only one has to be considered, the other blowing up exponentially at infinity. Here again the analysis of the classical trajectories helps to understand what happens. The modes that have to be rejected correspond to particles pushed to spatial infinity, i.e. quantum mechanically they describe fluctuations engendered by the Schwinger mechanism and disappear when $E \mapsto \infty$.

This indicates for the accelerated detector as well as for the charged black hole, that the Schwinger process first quickly switches off the external field, by emission of preferentially charged particles that will neutralize the plates of the condenser producing the electric field, or the black hole. Only then will the system reach a thermal (quasi)-equilibrium state at the Unruh/Hawking temperature.

- The ratio of population of a heavy two level charged detector accelerated and in interaction with the charged scalar field are in accordance with thermodynamics, as far as the charge difference between the two levels of the detector is small compared to their difference in masses. Moreover the amplitude of transition of such a detector can be obtained as the large mass limit of a model built on three interacting fields, a model where the levels of the detector are quantized. This shows that it is only in the limit where the recoil effects and pair creation *à la Schwinger* of the quanta describing the detector levels become negligible that the thermodynamical limit makes sense and is recovered. It is also noteworthy that, as expected, in the limit of electric field going to zero, the probabilities of transition by absorption or emission (7.21,7.23) of the fluctuating modes whose distributions become null (6.32) also vanish.

Acknowledgments

A Asymptotic behavior of Whittaker's functions

The Rindler's modes are built from the solutions of eq.(4.2), which are given by Whittaker's functions. For completeness, we recall in this appendix the few properties of these functions used in the main text.

Defining $\mathbf{W}^+ = \exp(a\xi)\mathcal{F}$ as a function of the imaginary variable $z = -i\epsilon\frac{qE}{2a^2}\exp(2a\xi)$, it is immediate to verify that $\mathbf{W}^+[z]$ has to satisfy the Whittaker's equation:

$$\left\{\frac{d^2}{dz^2} + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{(1-4\mu^2)}{4z^2}\right]\right\}\mathbf{W}^+ = 0 \qquad . \tag{A.1}$$

with

$$\kappa = \frac{i}{2}\left(\frac{\omega}{a} - \frac{m^2}{qE}\right) \qquad , \qquad \mu = i\frac{\omega}{2a} \tag{A.2}$$

The Whittaker's functions are related to the confluent hypergeometric function by

$$\mathbf{W}^+[z] = e^{-z/2}z^{1/2+\mu}\mathrm{F}[\kappa,\mu,z] \qquad , \tag{A.3}$$

where $\mathrm{F}[\kappa,\mu,z]$ is a solution of the Kummer's equation:

$$\left\{z\frac{d^2}{dz^2} + (1+2\mu - z)\frac{d}{dz} - (\frac{1}{2} + \mu - \kappa)\right\}\mathrm{F}[\kappa,\mu,z] = 0 \qquad . \tag{A.4}$$

In momentum space ($p = -\partial/\partial z$), this equation becomes first order. Solving it by an elementary quadrature and returning to the z representation, we obtain an integral representation of the Whittaker's function:

$$\begin{aligned}
\mathbf{W}^+[z] &= e^{-z/2}z^{\frac{1}{2}+\mu}\frac{1}{\Gamma[\frac{1}{2}+\mu-\kappa]}\int_0^\infty e^{-pz}p^{-\frac{1}{2}+\mu-\kappa}(1+p)^{-\frac{1}{2}+\mu+\kappa}dp \\
&= e^{-z/2}z^{\frac{1}{2}+\mu}\frac{z^{-2\mu}}{\Gamma[\frac{1}{2}+\mu-\kappa]}\int_0^\infty e^{-q}q^{-\frac{1}{2}+\mu-\kappa}(z+q)^{-\frac{1}{2}+\mu+\kappa}dq \qquad .
\end{aligned} \tag{A.5}$$

The second of these equations results from the change of variable $q = pz$ and the Jordan's lemma allowing to integrate on the real axis instead of the imaginary one. When z goes to infinity, eq.(A.5) gives immediately the asymptotic behavior of the function:

$$\mathbf{W}^+[z] \underset{z\to\infty}{\simeq} e^{-z/2}z^\kappa \qquad , \tag{A.6}$$

while near $z = 0$, the Whittaker's function behaves like

$$\mathbf{W}^+[z] \underset{z\to 0}{\simeq} \left[\frac{\Gamma[2\mu]}{\Gamma[\frac{1}{2}+\mu-\kappa]}z^{1/2-\mu} + \frac{\Gamma[-2\mu]}{\Gamma[\frac{1}{2}-\mu-\kappa]}z^{1/2+\mu}\right] \qquad . \tag{A.7}$$

If, instead of the change of function (A.3), we use the one with the opposite sign of μ, setting:

$$\mathbf{W}^+[z] = e^{-z/2}z^{1/2-\mu}\mathrm{F}[\kappa,-\mu,z] \tag{A.8}$$

we obtain a different integral representation, but of the same function (see eq.(A.7)):

$$\mathrm{e}^{-z/2}z^{\frac{1}{2}-\mu}\frac{1}{\Gamma[\frac{1}{2}-\mu-\kappa]}\int_0^\infty \mathrm{e}^{-pz}p^{-\frac{1}{2}-\mu-\kappa}(1+p)^{-\frac{1}{2}-\mu+\kappa}dp = \mathbf{W}^+[z] \qquad . \tag{A.9}$$

Indeed, both representations are solutions of a second order differential equation and have the same asymptotic behaviors. In the mathematical literature, this function is usually denoted by: $\mathrm{W}_{\kappa,\mu}(z)$, and satisfies the relation :

$$\mathbf{W}^+[z] = \mathrm{W}_{\kappa,+\mu}(z) = \mathrm{W}_{\kappa,-\mu}(z) \qquad . \tag{A.10}$$

A second, independent, solution of eq.(A.1), is given by the complex conjugate of the first one :

$$\mathbf{W}^-[z] = \left[\mathbf{W}^+[z]\right]^* \qquad . \tag{A.11}$$

So, we obtain complete sets of unnormalized modes, solutions of the wave equation (4.1):

$$
\begin{aligned}
\mathcal{W}_\epsilon^+(\tau,\xi) &= \frac{e^{-i\omega\tau}}{\sqrt{2\pi}}e^{-a\xi}W_{+i(\frac{\omega}{2a}-\frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-i\epsilon\frac{qE}{2a}e^{2a\xi}\right] \qquad , \\
\mathcal{W}_\epsilon^-(\tau,\xi) &= \frac{e^{-i\omega\tau}}{\sqrt{2\pi}}e^{-a\xi}W_{-i(\frac{\omega}{2a}-\frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[+i\epsilon\frac{qE}{2a}e^{2a\xi}\right] \qquad .
\end{aligned}
\tag{A.12}
$$

From the asymptotic expansions of the function $\mathbf{W}^\pm$, we read immediately the various coefficient that define the charges carried by these modes:

$$C_+(\mathcal{W}_\epsilon^+) = (C_-(\mathcal{W}_\epsilon^-))^* = \sqrt{2a}\left(\frac{qE}{2a^2}\right)^{[\frac{1}{2}+\frac{i}{2}(\frac{\omega}{a}-\frac{m^2}{qE})]}e^{\epsilon\frac{\pi}{4}(\frac{\omega}{a}-\frac{m^2}{qE})} \tag{A.13}$$

$$C_-(\mathcal{W}_\epsilon^+) = C_+(\mathcal{W}_\epsilon^-) = 0 \tag{A.14}$$

$$D_+(\mathcal{W}_\epsilon^+) = (D_-(\mathcal{W}_\epsilon^-))^* = \sqrt{2|\omega|}\left(\frac{qE}{2a^2}\right)^{[\frac{1}{2}+i\frac{\omega}{2a}]}e^{-i\epsilon\frac{\pi}{4}}e^{\epsilon\frac{\pi\omega}{4a}}\frac{\Gamma[-i\frac{\omega}{a}]}{\Gamma[\frac{1}{2}-i(\frac{\omega}{a}-\frac{m^2}{2qE})]} \tag{A.15}$$

$$D_-(\mathcal{W}_\epsilon^+) = (D_+(\mathcal{W}_\epsilon^-))^* = \sqrt{2|\omega|}\left(\frac{qE}{2a^2}\right)^{[\frac{1}{2}-i\frac{\omega}{2a}]}e^{-i\epsilon\frac{\pi}{4}}e^{-\epsilon\frac{\pi\omega}{4a}}\frac{\Gamma[i\frac{\omega}{a}]}{\Gamma[\frac{1}{2}+\frac{im^2}{2qE}]} \tag{A.16}$$

whose squared modulus are

$$|C_+(\mathcal{W}_\epsilon^+)|^2 = |C_-(\mathcal{W}_\epsilon^-)|^2 = \left(\frac{qE}{a}\right)e^{\epsilon\frac{\pi}{2}(\frac{\omega}{a}-\frac{m^2}{qE})} \tag{A.17}$$

$$|D_+(\mathcal{W}_\epsilon^+)|^2 = |D_-(\mathcal{W}_\epsilon^-)|^2 = \left(\frac{qE}{a}\right)e^{\epsilon\frac{\omega\pi}{2a}}\frac{\cosh[\pi(\frac{\omega}{a}-\frac{m^2}{2qE})]}{\left|\sinh[\pi\frac{\omega}{a}]\right|} \tag{A.18}$$

$$|D_-(\mathcal{W}_\epsilon^+)|^2 = |D_+(\mathcal{W}_\epsilon^-)|^2 = \left(\frac{qE}{a}\right)\frac{e^{-\epsilon\frac{\omega\pi}{2a}}\cosh\frac{\pi}{2}\frac{m^2}{qE}}{\left|\sinh[\pi\frac{\omega}{a}]\right|} \tag{A.19}$$

and satisfied the Wronskian relation:

$$\mathrm{sgn}(\omega)\left[|D_+(\mathcal{W}_\epsilon^+)|^2 - |D_-(\mathcal{W}_\epsilon^+)|^2\right] = \epsilon|C_+(\mathcal{W}_\epsilon^+)|^2 \qquad . \tag{A.20}$$

In the main text, we also make use of linear combinations of the previous Whittaker's function whose behaviors near $\xi = -\infty$, $[z = 0]$ are particularly simple. They are the functions $M_{\pm\kappa,\mu}(\pm z)$ and its complex conjugate $M_{\mp\kappa,-\mu}(\mp z)$:

$$M_{\kappa,\mu}(z) = \epsilon \left(\frac{\omega}{a}\right) \Gamma[i\frac{\omega}{a}] \left[\frac{e^{\epsilon\frac{\pi m^2}{2qE}}}{\Gamma\left[\frac{1}{2} + i(\frac{\omega}{a} - \frac{m^2}{2qE})\right]} \mathbf{W}^+ - \frac{e^{-i\epsilon\frac{\pi}{2}} e^{-\epsilon\frac{\pi}{2}(\frac{\omega}{a} - \frac{m^2}{qE})}}{\Gamma\left[\frac{1}{2} + i\frac{m^2}{2qE})\right]} \mathbf{W}^- \right] \quad , \qquad \text{(A.21)}$$

obeying the relations

$$\begin{aligned} M_{\kappa,\mu}(z) &= e^{-i\epsilon\frac{\pi}{2}} e^{+\epsilon\frac{\pi\ omega}{2a}} M_{-\kappa,\mu}(-z) \quad , \\ M_{\kappa,-\mu}(z) &= e^{-i\epsilon\frac{\pi}{2}} e^{-\epsilon\frac{\pi\omega}{2a}} M_{-\kappa,-\mu}(-z) \quad , \end{aligned} \qquad \text{(A.22)}$$

from which we have defined the (unnormalized) modes:

$$\mathcal{M}_\epsilon^+(\tau,\xi) = \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} e^{-a\xi} M_{+i(\frac{\omega}{2a} - \frac{m^2}{2qE}),i\frac{\omega}{2a}}\left[-i\epsilon\frac{qE}{2a}e^{2a\xi}\right] \quad , \qquad \text{(A.23)}$$

$$\mathcal{M}_\epsilon^-(\tau,\xi) = \frac{e^{-i\omega\tau}}{\sqrt{2\pi}} e^{-a\xi} M_{-i(\frac{\omega}{2a} - \frac{m^2}{2qE}),-i\frac{\omega}{2a}}\left[+i\epsilon\frac{qE}{2a}e^{2a\xi}\right] \quad , \qquad \text{(A.24)}$$

whose charge content is given by the coefficients:

$$|D_+(\mathcal{M}_\epsilon^+)|^2 = |D_-(\mathcal{M}_\epsilon^-)|^2 = |\omega| \left(\frac{qE}{2a^2}\right) e^{\epsilon\frac{\pi\omega}{2a}} \qquad \text{(A.25)}$$

$$|D_-(\mathcal{M}_\epsilon^+)|^2 = |D_+(\mathcal{M}_\epsilon^-)|^2 = 0 \qquad \text{(A.26)}$$

$$|C_+(\mathcal{M}_\epsilon^+)|^2 = |C_-(\mathcal{M}_\epsilon^-)|^2 = \left(\frac{\omega}{a}\right)\left(\frac{qE}{a}\right) \frac{e^{\epsilon\frac{\pi}{2}\left(\frac{\omega}{a} + \frac{m^2}{qE}\right)} \cosh\left[\pi\left(\frac{\omega}{a} - \frac{m^2}{2qE}\right)\right]}{\sinh\left[\frac{\pi\omega}{a}\right]}$$

$$\text{(A.27)}$$

$$|C_-(\mathcal{M}_\epsilon^+)|^2 = |C_+(\mathcal{M}_\epsilon^-)|^2 = \left(\frac{\omega}{a}\right)\left(\frac{qE}{a}\right) \frac{e^{-\epsilon\frac{\pi}{2}\left(\frac{\omega}{a} - \frac{m^2}{qE}\right)} \cosh\left[\pi\frac{m^2}{2qE}\right]}{\sinh\left[\frac{\pi\omega}{a}\right]}$$

$$\text{(A.28)}$$

verifying the Wronskian relation:

$$\text{sgn}(\omega) |D_+(\mathcal{M}_\epsilon^+)|^2 = \epsilon \left(|C_+(\mathcal{M}_\epsilon^+)|^2 - |C_-(\mathcal{M}_\epsilon^+)|^2\right) \qquad . \qquad \text{(A.29)}$$

Let us remark that $\mathcal{W}_\epsilon^+(\tau,\xi) = [\mathcal{W}_\epsilon^+(-\tau,\xi)]^*$ and $\mathcal{M}_\epsilon^+(\tau,\xi) = [\mathcal{M}_\epsilon^+(-\tau,\xi)]^*$, i.e. *in* and *out* classes of modes are related by a τ inversion followed by a complex conjugation.

B Explicit forms of some Bogoljubov coefficients

For sake of completeness we recall here the basic relations between Fock basis defined trough Bogoljubov transformations and give here the explicit expressions of the various Bogoljubov

coefficients that we have used in the main text. Suppose a quantum field operator $\hat{\Psi}$, satisfying a second order field equation, is defined in terms of two sets of linearly independent solutions ϕ_k^p, ϕ_k^{a*} and Φ_K^p, Φ_K^{a*}, labeled by the indices k and K, of the corresponding classical equation. We have

$$\hat{\Psi} = \sum_k a_i \phi_k^p + b_i^\dagger \phi_k^{a*} \tag{B.1}$$

$$= \sum_K A_K \Phi_K^p + B_K^\dagger \Phi_K^{a*} \quad , \tag{B.2}$$

and

$$\begin{cases} \Phi_K^p = \sum_j \alpha_K^j \phi_j^p + \beta_K^j \phi_j^{a*} \\ \Phi_K^{a*} = \sum_j \gamma_K^j \phi_j^p + \epsilon_K^j \phi_j^{a*} \end{cases} \quad \text{and} \quad \begin{cases} \phi_k^p = \sum_j \alpha_K^{j*} \Phi_K^p - \gamma_K^{j*} \Phi_K^{a*} \\ \phi_k^{a*} = \sum_j \epsilon_K^{j*} \Phi_K^{a*} - \beta_K^{j*} \Phi_K^p \end{cases} \tag{B.3}$$

If, as usual, we suppose these basis orthonormal, unitarity implies :

$$\begin{cases} \delta_{K\,K'} = \sum_j \alpha_{K'}^{j*} \alpha_K^j - \beta_{K'}^{j*} \beta_K^j \\ \delta_{K\,K'} = \sum_j \epsilon_{K'}^{j*} \epsilon_K^j - \gamma_{K'}^{j*} \gamma_K^j \\ 0 = \sum_j \gamma_{K'}^{j*} \alpha_K^j - \epsilon_{K'}^{j*} \beta_K^j \end{cases} \quad \begin{cases} \delta_{k\,k'} = \sum_J \alpha_J^{k*} \alpha_J^{k'} - \gamma_J^{k*} \gamma_J^{k'} \\ \delta_{K\,K'} = \sum_J \epsilon_J^{k'*} \epsilon_J^k - \beta_J^{k'*} \beta_J^k \\ 0 = \sum_J \gamma_J^{k'*} \epsilon_J^k - \alpha_J^{k'*} \beta_J^k \end{cases} \tag{B.4}$$

Combining these relations we obtain the links between the various creation and annihilation operators :

$$\begin{cases} a_j = \sum_K A_K \alpha_K^j + B_K^\dagger \gamma_K^j \\ b_j^\dagger = \sum_K A_K \beta_K^j + B_K^\dagger \epsilon_K^j \end{cases} \quad \text{and similarlly} \quad \begin{cases} A_K = \sum_j a_j \alpha_K^{j*} - b_j^\dagger \beta_K^{j*} \quad , \\ B_K^\dagger = \sum_K A_K \beta_K^j + B_K^\dagger \epsilon_K^j \quad , \end{cases} \tag{B.5}$$

For each set of operator is associate a "vacuum" state $|\Omega>$ and $|\omega>$ such that for all quantum number k and K :

$$A_K|\Omega >= B_K|\Omega >= 0 \quad \text{and} \quad a_k|\omega >= b_k|\omega >= 0 \quad . \tag{B.6}$$

A standard computation [30, 8] gives the link between these vacuum states :

$$|\Omega >= \mathcal{N} e^{\{\sum_{kl} m^{kl} a_k^\dagger b_l^\dagger\}} |\omega > \quad , \quad |\omega >= \mathcal{N}^* e^{\{\sum_{KL} M^{KL} A_K^\dagger B_L^\dagger\}} |\Omega > \tag{B.7}$$

where the matrices m^{kl} and M^{KL} are given by :

$$m^{kl} = \sum_J (\alpha^{-1})_J^k \beta_J^l \quad \text{and} \quad M^{KL} = \sum_j \gamma_L^j (\alpha^{-1})_K^j \quad . \tag{B.8}$$

The normalization coefficients are of particular physical interest; they give the projection of one vacuum state on the other (a fact that we anticipate in the notation $\mathcal{N}^*$). In the main text we only encounter "diagonal" Bogoljubov transformations, i.e. such that the two sets of indices $\{k\}$ and $\{K\}$ are identical and the various matrices α_K^k, ... $\epsilon_K^k \propto \delta_K^k$ i.e. diagonal. In such cases the matrices m^{kl} and M^{KL} are also diagonal and the normalization coefficient are particularly easy to evaluate. By factorizing the vacuum state according to the quantum number k: $|\omega >= \prod_k |\omega_k >$, we obtain :

$$1 = |\mathcal{N}|^2 \prod_k \sum_{n=0}^\infty \frac{1}{(n!)^2} <\omega_k| (m_{kk}^* a_k b_k)^n \left(m_{kk} a_k^\dagger b_k^\dagger\right)^n |\omega_k >= \prod_k \left(1 - |m_{kk}|^2\right)^{-1} = \prod_k |\alpha_k^k|^2 \quad , \tag{B.9}$$

the phase of $\mathcal{N}$ remaining arbitrary, the vacuum state being actually ray in the Hilbert space.

Bogoljubov transformations between Rindler *in* and *out* modes

On quadrant **R** , the coefficients connecting Rindler *in* and *out*-modes, coefficients occurring in eq.(4.26), are given by:

$$
\alpha_{\mathcal{U}\mathcal{U}}^{R}(\omega > 0) = \beta_{\mathcal{U}\mathcal{U}}^{R}(\,omega < 0) = \frac{\Gamma[\frac{1}{2} - i(\frac{\omega}{a} - \frac{m^2}{2qE})]}{\Gamma[-i\frac{\omega}{a}]} \frac{\mathcal{N}(\mathcal{U}_{out,R}^{\omega})}{\mathcal{N}(\mathcal{U}_{in,R}^{\omega})} \quad ,
$$

$$
\alpha_{\mathcal{U}\mathcal{V}}^{R}(\omega) = e^{i\frac{\pi}{2}} e^{-\frac{\pi\omega}{2a}} \frac{\Gamma[i\frac{\omega}{a}]}{\Gamma[-i\frac{\omega}{a}]} \frac{\Gamma[\frac{1}{2} - i(\frac{\omega}{a} - \frac{m^2}{2qE})]}{\Gamma[\frac{1}{2} + i\frac{m^2}{2qE}]} \frac{\mathcal{N}(\mathcal{U}_{out,R}^{\omega})}{\mathcal{N}(\mathcal{V}_{in,R}^{\omega})} \quad ,
$$

$$
\alpha_{\mathcal{V}\mathcal{U}}^{R}(\omega > 0) = \epsilon_{\mathcal{V}\mathcal{U}}^{R}(\omega < 0) = e^{i\frac{\pi}{2}} e^{-\frac{\pi\omega}{2a}} \frac{\Gamma[\frac{1}{2} - i(\frac{\omega}{a} - \frac{m^2}{2qE})]}{\Gamma[\frac{1}{2} - i\frac{m^2}{2qE}]} \frac{\mathcal{N}(\mathcal{V}_{out,R}^{\omega})}{\mathcal{N}(\mathcal{U}_{in,R}^{\omega})} \quad ,
$$

$$
\epsilon_{\mathcal{V}\mathcal{V}}^{R}(\omega < 0) = \gamma_{\mathcal{V}\mathcal{V}}^{R}(\omega > 0) = \left(\frac{a}{\omega}\right) e^{-\frac{\pi m^2}{2qE}} \frac{\Gamma[\frac{1}{2} - i(\frac{\omega}{a} - \frac{m^2}{2qE})]}{\Gamma[-i\frac{\omega}{a}]} \frac{\mathcal{N}(\mathcal{V}_{out,R}^{\omega})}{\mathcal{N}(\mathcal{V}_{in,R}^{\omega})} \quad .
$$

On quadrant **P** , see eq.(4.54) they are given by:

$$
\alpha_{\mathcal{U}\mathcal{V}}^{P}(\omega > 0) = \gamma_{\mathcal{U}\mathcal{V}}^{P}(\omega < 0) = e^{\pi\left(\frac{\omega}{2a} - \frac{m^2}{2qE}\right)} \frac{\Gamma[1 + i\frac{\omega}{a}]}{\Gamma[\frac{1}{2} + i\frac{m^2}{2qE}]} \frac{\mathcal{N}(\mathcal{U}_{out,P}^{\omega})}{\mathcal{N}(\mathcal{V}_{in,P}^{\omega})} \quad ,
$$

$$
= \beta_{\mathcal{V}\mathcal{U}}^{P\,*}(\omega < 0) = \epsilon_{\mathcal{V}\mathcal{U}}^{P\,*}(\omega > 0) \tag{B.10}
$$

$$
\beta_{\mathcal{U}\mathcal{U}}^{P}(\omega > 0) = \epsilon_{\mathcal{U}\mathcal{U}}^{P}(\omega < 0) = e^{i\frac{\pi}{2}} e^{-\pi\frac{m^2}{2qE}} \frac{\Gamma[1 + i\frac{\omega}{a}]}{\Gamma[\frac{1}{2} + i(\frac{\omega}{a} - \frac{m^2}{2qE})]} \frac{\mathcal{N}(\mathcal{U}_{out,P}^{\omega})}{\mathcal{N}(\mathcal{U}_{in,P}^{\omega})} \quad ,
$$

$$
= \alpha_{\mathcal{V}\mathcal{V}}^{P\,*}(\omega < 0) = \gamma_{\mathcal{V}\mathcal{V}}^{P\,*}(\omega > 0) \quad . \tag{B.11}
$$

Bogoljubov transformation between Unruh and Rindler *out* modes

In the **R** and **L** sectors, Unruh *out* modes are related to Rindler modes as:

$$
\begin{aligned}
\Omega_{\omega}^{p\,out} &= \theta(-\omega)\left\{\beta_{\Omega\mathcal{V}_R}^{out}(\omega)\mathcal{V}_{\omega,R}^{out} + \alpha_{\Omega\mathcal{U}_L}^{out}(\omega)\mathcal{U}_{\omega,L}^{out}\right\} + \\
&\qquad \theta(\omega)\left\{\alpha_{\Omega\mathcal{V}_R}^{out}(\omega)\mathcal{V}_{\omega,R}^{out} + \beta_{\Omega\mathcal{U}_L}^{out}(\omega)\mathcal{U}_{\omega,L}^{out}\right\} \\
\varpi_{\omega}^{p\,out} &= \alpha_{\varpi\mathcal{U}_R}^{out}(\omega)\mathcal{U}_{\omega,R}^{out} \\
\Omega_{\omega}^{a\,out\,*} &= \theta(-\omega)\left\{\epsilon_{\Omega\mathcal{V}_R}^{out}(\omega)\mathcal{V}_{\omega,R}^{out} + \gamma_{\Omega\mathcal{U}_L}^{out}(\omega)\mathcal{U}_{\omega,L}^{out}\right\} + \\
&\qquad \theta(\omega)\left\{\gamma_{\Omega\mathcal{V}_R}^{out}(\omega)\mathcal{V}_{\omega,R}^{out} + \epsilon_{\Omega\mathcal{U}_L}^{out}(\omega)\mathcal{U}_{\omega,L}^{out}\right\} \\
\varpi_{\omega}^{a\,out\,*} &= \epsilon_{\varpi\mathcal{V}_L}^{out}(\omega)\mathcal{V}_{\omega,L}^{out},
\end{aligned} \tag{B.12}
$$

with

$$
\alpha_{\Omega\mathcal{U}_L}^{out}(\omega) = \beta_{\Omega\mathcal{U}_L}^{out}(\omega) = e^{i\frac{\pi}{2}} e^{-\frac{\pi m^2}{4qE}} \left\{\frac{\cosh\pi(\frac{m^2}{2qE} - \frac{\omega}{a})}{|\sinh\frac{\pi\omega}{a}|}\right\}^{\frac{1}{2}}
$$

$$
\alpha_{\Omega\mathcal{V}_R}^{out}(\omega) = \beta_{\Omega\mathcal{V}_R}^{out}(\omega) = e^{-\frac{\pi m^2}{4qE}} e^{\frac{\pi\omega}{2a}} \left\{\frac{\cosh\pi(\frac{\pi m^2}{2qE} - \frac{\omega}{a})}{|\sinh\frac{\pi\omega}{a}|}\right\}^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - i\frac{m^2}{2qE} + i\frac{\omega}{a})}{\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})}
$$

$$\alpha_{\varpi\mathcal{U}_R}^{out}(\omega) = \epsilon_{\varpi\mathcal{V}_L}^{out}(\omega) = \varepsilon(\omega)$$

$$\epsilon_{\Omega\mathcal{V}_R}^{out}(\omega) = \gamma_{\Omega\mathcal{V}_R}^{out}(\omega) = \alpha_{\Omega\mathcal{U}_L}^{out\,*}(\omega) = \beta_{\Omega\mathcal{U}_L}^{out\,*}(\omega)$$

$$\gamma_{\Omega\mathcal{U}_L}^{out}(\omega) = \epsilon_{\Omega\mathcal{U}_L}^{out}(\omega) = \alpha_{\Omega\mathcal{V}_R}^{out\,*}(\omega) = \beta_{\Omega\mathcal{V}_R}^{out\,*}(\omega). \tag{B.13}$$

In the same way, we get the expression of the Unruh *out* modes in terms of Rindler **P** and **F** modes

$$\Omega_\omega^{p\,out} = \alpha_{\Omega\mathcal{U}_F}^{out}(\omega)\mathcal{U}_{\omega,F}^{out} + \theta(\omega)\left\{\alpha_{\Omega\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{\omega,P}^{out} + \beta_{\Omega\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{\omega,P}^{out}\right\} +$$
$$\theta(-\omega)\left\{\beta_{\Omega\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{\omega,P}^{out} + \alpha_{\Omega\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{\omega,P}^{out}\right\} \tag{B.14}$$

$$\varpi_\omega^{p\,out} = \theta(\omega)\beta_{\varpi\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{\omega,P}^{out} + \theta(-\omega)\alpha_{\varpi\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{\omega,P}^{out} \tag{B.15}$$

$$\Omega_\omega^{a\,out\,*} = \epsilon_{\Omega\mathcal{V}_F}^{out}(\omega)\mathcal{V}_{\omega,F}^{out} + \theta(\omega)\left\{\epsilon_{\Omega\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{\omega,P}^{out} + \gamma_{\Omega\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{\omega,P}^{out}\right\} +$$
$$\theta(-\omega)\left\{\gamma_{\Omega\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{\omega,P}^{out} + \epsilon_{\Omega\mathcal{U}_P}^{out}(\omega)\mathcal{U}_{\omega,P}^{out}\right\} \tag{B.16}$$

$$\varpi_\omega^{a\,out\,*} = \theta(\omega)\gamma_{\varpi\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{\omega,P}^{out} + \theta(-\omega)\epsilon_{\varpi\mathcal{V}_P}^{out}(\omega)\mathcal{V}_{\omega,P}^{out}, \tag{B.17}$$

with coefficients:

$$\alpha_{\Omega\mathcal{U}_F}^{out}(\omega) = \epsilon_{\Omega\mathcal{V}_F}^{out}(\omega) = 1$$

$$\alpha_{\Omega\mathcal{U}_P}^{out}(\omega > 0) = \beta_{\Omega\mathcal{U}_P}^{out}(\omega < 0) = \epsilon_{\Omega\mathcal{V}_P}^{out\,*}(\omega > 0) = \gamma_{\Omega\mathcal{V}_P}^{out\,*}(\omega < 0) = \alpha_{\Omega\mathcal{V}_F}^{in\,*}(\omega > 0)$$

$$\beta_{\Omega\mathcal{V}_P}^{out}(\omega > 0) = \alpha_{\Omega\mathcal{V}_P}^{out}(\omega < 0) = \gamma_{\Omega\mathcal{U}_P}^{out\,*}(\omega > 0) = \epsilon_{\Omega\mathcal{U}_P}^{out\,*}(\omega < 0) = \alpha_{\Omega\mathcal{U}_F}^{in\,*}(\omega < 0)$$

$$\beta_{\varpi\mathcal{U}_P}^{out}(\omega > 0) = \alpha_{\varpi\mathcal{U}_P}^{out}(\omega < 0) = \epsilon_{\varpi\mathcal{V}_P}^{out\,*}(\omega < 0) = \gamma_{\varpi\mathcal{V}_P}^{out\,*}(\omega > 0) = \beta_{\varpi\mathcal{V}_F}^{in\,*}(\omega < 0). \tag{B.18}$$

C   Schwinger representation for wave function products

In subsection 5.1 we made use of several integral representations of products of modes. Hereafter, we briefly indicate how they are obtained.

•Inertial observer Using the relations (9.240) and (6.669.3) of ref. [13], we may express the product of modes: $\varphi_\sigma^{p\,out}(x)\varphi_\sigma^{a\,out}(x)$ as:

$$\varphi_\sigma^{p\,out}(x)\varphi_\sigma^{a\,out}(x) = -e^{-i\frac{\pi}{4}}\sqrt{\frac{\pi}{2}}\frac{2\sqrt{qE}}{|M|^2}2^{-i\frac{m^2}{2qE}}\frac{1}{\Gamma(\frac{1}{4} + i\frac{m^2}{4qE})\Gamma(\frac{3}{4} + i\frac{m^2}{4qE})}$$
$$\int_0^\infty \frac{d\xi}{(\sinh\xi)^{-\frac{1}{2}}}e^{iqE(z+\frac{\sigma}{qE})(\cosh\xi - \sinh\xi)}(\coth\frac{\xi}{2})^{-i\frac{m^2}{2qE}}. \tag{C.1}$$

Introducing the new variable s defined by $\sinh 2qEs = (\sinh\xi)^{-1}$ in this integral, we get:

$$\varphi_\sigma^{p\,out}(x)\varphi_\sigma^{a\,out}(x) = = -\frac{1}{\sqrt{2i\pi}}\frac{e^{-\frac{3\pi m^2}{4qE}}}{\sqrt{2\pi}}\frac{\sqrt{qE}}{\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})}\int_0^\infty \frac{ds}{(\sinh 2qEs)^{\frac{1}{2}}}e^{-im^2 s}e^{iqE(z+\frac{\sigma}{qE})^2(\coth 2qEs - \frac{1}{\sinh 2qEs})}. \tag{C.2}$$

•Accelerated observer In the same way, the product of wave functions $\mathcal{U}_\omega^{out}(x)\mathcal{V}_\omega^{in\,*}(x)$ can be expressed as the integral :

$$\mathcal{U}_\omega^{out}(x)\mathcal{V}_\omega^{in\,*}(x) = -i\frac{qE}{2}|\mathcal{N}(\mathcal{U}_\omega^{out})|^2\frac{1}{2\pi a^2}\frac{\left(\Gamma(1 + i\frac{\omega}{a})\right)^2}{\Gamma(\frac{1}{2} + i\frac{\omega}{a} - i\frac{m^2}{2qE})\Gamma(\frac{1}{2} + i\frac{m^2}{2qE})}$$

$$\int_{-\infty}^{\infty} \frac{d\xi}{\cosh\xi} e^{i(\frac{\omega}{a} - \frac{m^2}{qE})\xi} e^{i\frac{qE}{2}\rho^2 \tanh\xi} I_{i\frac{\omega}{a}}\left(\frac{qE\rho^2}{2\cosh\xi}\right) \tag{C.3}$$

thanks to the eq. (6.669.6) of [13].

References

[1] S. W. Hawking, *Commun. Math. Phys.* **43** (1975), 199.

[2] W. G. Unruh, *Phys. Rev. D* **14** (1976), 870.

[3] W. Heisenberg and H. Euler, *Z. Phys.* **98**, 714 (1936), 714.

[4] J. Schwinger, *Phys. Rev.* **82** (1951), 664.

[5] R. Brout, R. Parentani and Ph. Spindel, *Nucl. Phys.* **B353** (1991), 209.

[6] R. Parentani and S. Massar, *Phys. Rev. D* **55** (1997), 3603.

[7] Cl. Gabriel, S. Massar, R. Parentani, Ph. Spindel, *Phys.Rev.D* **57** (1998), 6496.

[8] R. Brout, S. Massar, R. Parentani and Ph. Spindel, *Phys. Rep.* **260** (1995), 329.

[9] R. Brout, S. Massar, S. Popescu, R. Parentani and Ph. Spindel, *Phys.Rev.D* **52** (1995), 1119.

[10] R. Brout, *Private communication.*

[11] N. D. Birrell, P. C. W. Davies, "Quantum fields in curved space", Cambridge University Press, Cambridge (2 ed.), 1984.

[12] R. Parentani and R. Brout, *Nucl. Phys.* **B388** (1992), 474.

[13] I.S. Gradshteyn, I.M. Ryzhik, "Table of integrals, series and products", Academic Press Inc., San Diego, 1980.

[14] R. Parentani, Ph. D. Thesis, Université Libre de Bruxelles, (1992).

[15] D. C. Khandekar, S. V. Lawande, *Phys. Rep.* **137** (1986), 115.

[16] R. Feynman, A. Hibbs, "Quantum Mechanics and path Integrals", McGrawHill Book Company, New York, 1965.

[17] W. Pauli, *Handb. Phys* **24** (1933), 120; B. S. DeWitt, *Phys. Rev.* **85** (1952), 653.

[18] J. H. Van Vleck, *P. Nat. Acad. of sciences* **14** (1928), 178; W. Pauli, "Ausgewahlte Kapitel der Feldquantisierung", Lecture Notes , Zurich (1952).

[19] C. De Witt-Morette, *Ann. Phys.* **97** (1976), 367.

[20] R. Parentani, *Nucl. Phys. B* **454** (1995), 227.

[21] B. Reznik, *First Order Corrections to the Unruh Effect* (1995) gr-qc/9511033.

[22] P. Grove, *Class. Quantum Grav.*, **3** (1986), 801.

[23] D. Raine, D. Sciama and P. Grove, *Proc. R. Soc.* (London) A **485** (1991), 205.

[24] S. Massar, R. Parentani, R. Brout, *Class. Quantum Grav.* **10** (1993), 385.

[25] Y. Choquet-Bruhat, C. De Witt-Morette, M. Dillard-Bleick, "Analysis, manifolds and Physics", NortHolland Pub., Amsterdam, 1977.

[26] S. Coleman, *in* "Th Whys of Subnuclear Physics" (A. Zichichi, Ed.) Plenum Press, New York, 1979.

[27] Ph. Spindel, "Mécanique analytique", vol.**2**, Presses universitaires montoises, Mons, 1998.

[28] W. Troost, H. Van Dam, *Phys. Lett B* **87** (1977), 149; *Nucl. Phys.* **B152** (1979), 442.

[29] R. Parentani, *Class. Quantum Grav.*, **10** (1993), 1409.

[30] Kamefuchi, Umezawa, *Nuovo Cimento* **31** (1964), 2.

[31] Ph. Spindel, *in* "The Gardener of Eden" (P. Nicoletopoulos and J. Orloff, Ed.) , Physicalia Magazine **12**, 207 , 1990.

[32] M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions", Dover Pub., New York, 1968.

[33] G. Gibbons, *Commun. Math. Phys.* **44** (1975), 245.

Figures captions

Fig. 1 The four Rindler patches and their coordinates.

Fig. 2 Typical trajectories of charged particles on Rindler patches: plain curves correspond to particles trajectories, the dashed one to an antiparticle trjectory. The numbers labelling the different trajectories refer to the discussion in the main text.

Fig. 3 Schematic representation, in the various Rindler quadrant, of the charged carried by the various modes defined in section **3**, and of typical wavepackets built out of them with the appropriate mean value of ω. On the **R** and **L** quadrants, we consider packets centered around a positive mean ω; on the **P** and **F** quadrants, the packets are built out of modes whose ω are essentially negative.

Fig. 4 *In* and *out* Unruh modes represented as superposition of Rindler modes and their charge contents

Fig. 5 Ratio of the total probabilities of excitation and deexcitation of a charged detector of mass gap $\Delta M/a = 0.1$, normalized with the Boltzman factor $\exp -[2\,\pi\,\omega_\star/a]$, as function of the mass m/a and "charge" qE/a^2 of the exchanged quantum.

Figures

i^+

$I^+_{F,L}$ F $I^+_{F,R}$

9 8

$I^+_{L,F}$

H^+_L H^+_R 2

$I^+_{R,F}$

L O 5 3 4 R 4' i^0_R

i^0_L

H^-_L H^-_R 1

$I^-_{L,P}$

$I^-_{R,P}$

7 6

$I^-_{P,L}$ $I^-_{P,R}$ P

i^-

p, out
Ω_ω

p, out
ω_ω

a, out
Ω_ω

a, out
ω_ω

p, in
Ω_ω

p, in
ω_ω

a, in
Ω_ω

a, in
ω_ω